Published by

Department of Mathematis
Faculty of Mathematics and Natural Sciences
Universitas Gadjah Mada
Sekip Utara, Yogyakarta, Indonesia
Telp. +62 (274) 552243, 7104933
Fax. +62 (274) 555131

ISBN : 978-979-17979-4-8

Proceedings The 7 SEAMS-UGM 2015

SEAMS UGM 2015

PROCEEDINGS
The 7th SEAMS-UGM 2015
International Conference On Mathematics and Its Applications

"Enhancing the Role of Mathematics in Interdisciplinary Researches"

August, 18th - 21st 2015

Editors
Fajar Adi Kusumo
Indah Emilia Wijayanti
Irwan Endrayanto Aluicius
Yeni Susanti

Depatment of Mathematics
Faculty of Mathematics and Natural Sciences
Universitas Gadjah Mada
2016

PROCEEDINGS OF
THE 7TH SOUTHEAST ASIAN MATHEMATICAL SOCIETY - UNIVERSITAS GADJAH MADA INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS 2015

"Enhancing the Role of Mathematics in Interdisciplinary Researches"

Yogyakarta, Indonesia, August 18th – 21st, 2015

DEPARTMENT OF MATHEMATICS
FACULTY OF MATHEMATICS AND NATURAL SCIENCES
UNIVERSITAS GADJAH MADA
YOGYAKARTA, INDONESIA
2016

Published by

Department of Mathematics
Faculty of Mathematics and Natural Sciences
Universitas Gadjah Mada
Sekip Utara, Yogyakarta, Indonesia
Telp. +62 (274) 7104933, 552243
Fax. +62 (274) 555131

PROCEEDINGS OF
THE 7TH SOUTHEAST ASIAN MATHEMATICAL SOCIETY-UNIVERSITAS GADJAH MADA INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS 2015

ISBN 978-979-17979-4-8

PROCEEDINGS OF THE 7TH SOUTHEAST ASIAN MATHEMATICAL SOCIETY-UNIVERSITAS GADJAH MADA INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS 2015

EDITORIAL BOARDS

LIST OF REVIEWERS

Abdurakhman	Universitas Gadjah Mada
Agus Maman Abadi	Universitas Negeri Yogyakarta
Agus Suryanto	Universitas Brawijaya
Andrei Sleptchenko	Qatar University
Ari Suparwanto	Universitas Gadjah Mada
Atok Zulijanto	Universitas Gadjah Mada
Budi Surodjo	Universitas Gadjah Mada
Ch. Rini Indrati	Universitas Gadjah Mada
Danardono	Universitas Gadjah Mada
Dellavitha Nasution	Institut Teknologi Bandung
Dhoriva Urwatul Wutsqa	Universitas Negeri Yogyakarta
Diari Indriati	Universitas Negeri Surakarta
Fajar Adi Kusumo	Universitas Gadjah Mada
Fatmawati	Universitas Airlangga
Gunardi	Universitas Gadjah Mada
Hasmawati	Universitas Hasanuddin
Hengki Tasman	Universitas Indonesia
Herni Utami	Universitas Gadjah Mada
Herry Pribawanto Suryawan	Universitas Sanata Dharma
I Nyoman Budiantara	Institut Teknologi Sepuluh Nopember
I Wayan Sudarsana	Universitas Tadulako
Imam Solekhudin	Universitas Gadjah Mada
Indah Emilia Wijayanti	Universitas Gadjah Mada
Indarsih	Universitas Gadjah Mada
Intan Muchtadi-Alamsyah	Institut Teknologi Bandung
Irwan Endrayanto	Universitas Gadjah Mada
Isnaini Rosyida	Universitas Negeri Semarang
Khurul Wardati	UIN Sunan Kalijaga
L.H. Wiryanto	Institut Teknologi Bandung
Lina Aryati	Universitas Gadjah Mada
Manuharawati	Universitas Negeri Surabaya
Marcellinus Andy Rudhito	Universitas Sanata Dharma
MHD. Reza M.I. Pulungan	Universitas Gadjah Mada
Muhammad Wakhid Musthofa	UIN Sunan Kalijaga
Nor Haniza Sarmin	Universiti Teknologi Malaysia
Nurdin	Universitas Hasanuddin
Nurhayadi	Universitas Tadulako
Nursanti Anggriani	Universitas Padjadjaran
Salmah	Universitas Gadjah Mada
Sri Wahyuni	Universitas Gadjah Mada
Sudi Mungkasi	Universitas Sanata Dharma
Suhartono	Institut Teknologi Sepuluh Nopember
Sumardi	Universitas Gadjah Mada
Supama	Universitas Gadjah Mada
Syafrizal Sy	Universitas Andalas
Windarto	Universitas Airlangga
Yansen Marpaung	Universitas Sanata Dharma
Yeni Susanti	Universitas Gadjah Mada

PREFACE

First of all, we would like to thank to all participants of "The 7th SEAMS UGM 2015 International Conference on Mathematics and Its Applications" which was held on August 18-21, 2015. The conference is very important as a communication forum of the mathematicians, not only in Indonesia, but also in Southeast Asia and surrounding areas. We also thank to the steering committee members and all of the reviewers for all supports during the conference and the preparation of this proceedings.

As the scientific documentation of the conference, we provide two-types of procedings. The first one is the AIP Proceedings which contains the high quality paper selected by blind review process. The second one is the regular proceedings, which contain the selected papers which are not published in the AIP Proceedings and the paper of our invited speakers.

We would like to say thanks to all authors who have submitted the paper to our proceedings. During the review process, we found that almost all papers has good quality. However due to the limitation number of the paper which can be published in our proceedings, we should select the submitted paper based on the reviewer recommendation and score. So, there are some papers which are not accepted to publish in this proceedings, we apologize to the authors about this inconvenience.

Lastly, we would like to say thanks for all partners and all sponsors in supporting our conference.

Warm regards,

Dr. Fajar Adi Kusumo
Editor in Chief

CONTENTS

Proceedings of "The 7^{th} SEAMS-UGM Conference 2015"
Topic, pp. 1–23.

Hedging in Interval Models: A Case Study[1]

S.A.M. Dooper[*] and J.C. Engwerda[‡]

[*]*Tilburg University, The Netherlands*
[†]*Tilburg University, Dept. of Econometrics and O.R. P.O. Box: 90153, 5000 LE Tilburg, The Netherlands; e-mail: engwerda@uvt.nl*

Abstract. The interval model is a model introduced in mathematical finance to determine the price range of options where the general setting is deterministic instead of stochastic. In this paper, we give a short introduction on the interval model and provide an overview of some important properties of it. In the second part of this paper we use this interval model to compare the price range obtained using the interval model for several European call options to the so called Bid-Ask prices reported on the stock exchange for these options.

Keywords: option pricing, interval model, robust control.

INTRODUCTION

Many mathematical models used in finance nowadays use a stochastic approach to determine option prices. These stochastic models focus on the aspect of risk. In most models, a normal distribution is assumed to simulate the movements of a stock because of its simplicity and statistical properties. One problem in practice, however, is that a normal distribution does not always provide a correct reflection of reality. It undervalues extreme movements that can occur during volatile periods on stock markets. Another problem is that in some cases it is not possible to come up with a distribution. Just because there is no data or past experience available (think of new products that have no customers yet). When such situations occur, one must be careful with using the available option pricing formulas. In those cases the conditions under which these formulas can be used are usually not met. In such cases a model just assuming that future stock prices may fluctuate between some a priori determined bounds, may be more appropriate. And, if such a model applies, it seems reasonable to assume that both buyers and sellers of an option will use then a worst-case scenario realization perspective of prices to determine a, in their eyes, fair price of the option.

This leads to the interval market model approach for pricing of options. The development of this theory was initiated at the end of the 20^{th} century and main results were recently collected and published in [5]. Like in conventional option pricing models it assumes that the market does not allow arbitrage opportunities.

This interval market model approach can be seen as an additional instrument in coping with market uncertainty. Additional to well-known ways to assess portfolio risk, like: measuring the value-at-risk ([9], [10]), stress testing ([11]) and uncertain volatility modeling ([2]).

In this paper we will review in Section 3 some important results from this interval market model concerning the calculation of a price range for a European call option, its so-called fair price interval (FPI). More details and proofs on this can be found in Chapters 3 and 4 of [5]. To calculate this interval, we use two specific hedging strategies that lead to different trading cost scenarios: the so-called stop-loss and delta hedging strategy, respectively. We will briefly outline these strategies in Section 2. These hedging strategies generate the lowest worst-case costs and the highest best-case costs that can be obtained for the option price in this interval market model. In Sections 4 and 5 we present some preliminary work for the application Section 6. We discuss some extensions of the basic algorithm to calculate the FPI, introduced in Section 3.1, and some parameter estimation issues, respectively. Next, in Section 6, we confront

[1] This paper is based on the Bachelor Thesis [3] written by S.A.M. Dooper.
[2] Corresponding Author

theory with practice. In this section we perform a first study to answer the question how well the interval market model performs in practice. In more specific terms, we are interested in whether the FPI obtained by the interval market model gives a good estimate for the Bid-Ask spread reported for options on the stock exchange. In order to test this model in practical situations, we compute the FPI for three European call options with different volatility trends. We use three types of volatility estimation methods that are widely used for their simplicity, accuracy or for their theoretical value: the implied volatility, the historical volatility and the GARCH(1,1) model. The paper ends with a number of observations from the case study and items for further research.

PRELIMINARIES

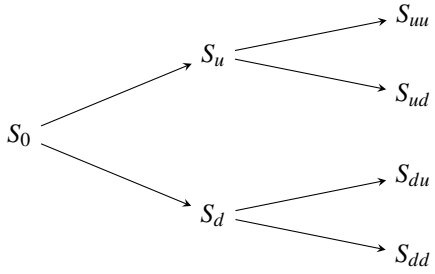
Before we introduce in Section 3 the interval market model, together with some important properties, we introduce in this section two trading strategies that play an important role in determining the so-called fair price interval of a European call option. We introduce these two trading strategies, the stop-loss and delta strategy, within the most simple setting: the binomial tree model. The binomial tree model is a well known model for option pricing. This model uses a discrete-time setting. Let intermediate time points be denoted by t_j, $j = 0,1,2,\ldots,N$, and the price of the asset be denoted by S_j at time t_j. An *asset price path* is defined then as a sequence:

$$\mathscr{S} = \{S_0,\ldots,S_N.\} \tag{1}$$

We assume that the initial asset price, S_0, is known, as well as the time horizon t_N, and time to maturity T. The number of intermediate trading-time steps is denoted by N. Furthermore, we assume that we either have an upward movement or a downward movement at all points in time $t_j = 0,1,2,\ldots,N-1$, denoted by u_j and d_j respectively, and that there exist no arbitrage opportunities in the market. We can now define the binomial tree model $\mathbb{B}^{u,d}$ as:

$$\mathbb{B}^{u,d} := \{\mathscr{S} | S_{j+1} \in \{d_j S_j, u_j S_j\} \text{ for } j = 0,1,2,\ldots,N-1\}. \tag{2}$$

The basic idea of the binomial tree model is visualized in the figure, below, for N = 2, where the jump factors u_j and d_j are assumed to be the same for all j:



Binomial Tree $\mathbb{B}^{u,d}$.

Stop-loss strategy

One of the easiest hedging strategies a trader on the stock market can apply is the stop-loss strategy. The idea is to buy 1 unit of the stock when the stock price S is higher than the strike price X of the option, and to sell 1 unit of the stock as soon as it is below X. By doing this the trader will make sure that he will own the stock at expiration time T if the stock price is larger than X. In terms of strategies, the stop-loss strategy is denoted as follows:

$$g_j(S_j) = \begin{cases} 0, & \text{if } S_j \leq X \\ 1, & \text{if } S_j > X \end{cases}$$

Here, the value of the function g denotes what position to hold in the asset, where a value of 1 means to hold a positive

(long) position in the asset, and a value of 0 means to sell the asset. If we consider a trader who has written a call option, then we have next result about the involved trading cost (see, e.g., [5, Theorem 3.1], where $[K]^+ := \max(K,0)$.

Theorem 1 *The total costs of hedging and closure (see Definition 2 for a formal definition of this cost) for a call option using a stop-loss strategy is* $Q^{stop-loss}(S_0) = [S(0) - X]^+$.

Delta strategy

The delta trading strategy (or delta hedging) is an active trading strategy used in the binomial tree model. It is an active strategy in the sense that the trader has to evaluate his holdings and rebalance his portfolio at every time step t_j. The main theoretical advantage of applying the delta strategy is that the trader can hedge his portfolio in such a way that the payoff at the time of maturity T becomes the same for all nodes at time t_N.

To demonstrate how this trading strategy works, we consider a trader who sold short one European call option at time t_0 with strike price X and owns a fraction Δ_0 of the asset. Let $F_j(S_j)$ denote the value of the option at time t_j with asset price S_j. Furthermore, assume that the trader has a long position of Δ_j shares of the asset and a short position in the option contract. At this point, the trader who has sold short the call option has the obligation to pay the buyer the value of the call option at the time of expiration T. This means that the value of the trader's portfolio at time t_{j+1} if the stock price moves up/down is

$$\Delta_j u_j S_j - F_{j+1}(u_j S_j) \text{ if } S_{j+1} = u_j S_j, \text{ and } \Delta_j d_j S_j - F_{j+1}(d_j S_j) \text{ if } S_{j+1} = d_j S_j, \text{ respectively.}$$

From these 2 equations we can choose Δ_j as follows:

$$S_j u_j \Delta_j - F_{j+1}(u_j S_j) = S_j d\Delta_j - F_{j+1}(d_j S_j) \text{ or } \Delta_j = \frac{1}{S_j} \frac{F_{j+1}(u_j S_j) - F_{j+1}(d_j S_j)}{u_j - d_j}. \tag{3}$$

Equation (3) tells us that we should choose Δ_j as the ratio of the change in the option price to the change in the stock price. In this case, the portfolio is riskless and must therefore earn the risk-free interest rate, denoted by r_j. Let the time elapsed between t_{j+1} and t_j be denoted by Δt_j. Then the present value of the portfolio at time t_j equals

$$(\Delta_j u_j S_j - F_{j+1}(u_j S_j))\mathrm{e}^{-r_j \Delta t_j}.$$

Considering we know the portfolio earns the risk free rate r_j, the present value of the portfolio must equal $\Delta_j S_j - F_j(S_j)$. This leads to the following equation

$$(\Delta_j u_j S_j - F_{j+1}(u_j S_j))\mathrm{e}^{-r_j \Delta t_j} = \Delta_j S_j - F_j(S_j).$$

Substituting equation (3) for Δ_j gives the following backward recursion formula for F_j:

$$\begin{aligned} F_j(S_j) =& \frac{F_{j+1}(u_j S_j)(1 - d_j \mathrm{e}^{-r_j \Delta t_j}) + F_{j+1}(d_j S_j)(u_j \mathrm{e}^{-r_j \Delta t_j} - 1)}{u_j - d_j} \\ =& q_j F_{j+1}(u_j S_j) + (\mathrm{e}^{-r_j \Delta t_j} - q_j) F_{j+1}(d_j S_j), \text{with} \\ F_N(S_N) =& [S_N - X]^+, \text{ where } q_j := \frac{1 - \mathrm{e}^{-r_j \Delta t_j}}{u_j - d_j}. \end{aligned}$$

From this recursion formula, we can also derive a recursive formula for the corresponding delta-hedging strategy (see, e.g., [7] or [5]):

$$\Delta_j(S_j) = \lambda_j \Delta_{j+1}(u_j S_j) + (1 - \lambda_j)\Delta_{j+1}(d_j S_j),$$

where $\lambda_j = u_j q_j$ and $\Delta_{N-1}(S_{N-1}) = \frac{[u_{N-1} S_{N-1} - X]^+ - [d_{N-1} S_{N-1} - X]^+}{(u_{N-1} - d_{N-1}) S_{N-1}}$.
So, summarizing, using this trading strategy the trader's value of the portfolio does not change over time. He can avoid any risk over time using this trading strategy.
For simplicity we will assume from now on that u_j and d_j are the same for all specific time moments t_j. So we can just write u and d from now on. Furthermore we assume that $d < 1 < u$.

THE INTERVAL MARKET MODEL

We again consider a discrete-time framework similar to the binomial tree model. The only main difference now is that the asset price S_j can fluctuate between an upper bound and a lower bound, uS_j and dS_j, respectively. For simplicity assume, for this moment, that the risk-free interest rate $r_j = 0$. Next definitions, corollaries and theorems are obtained from [5].
The *interval model* assumes that future stock prices may fluctuate as follows with its current price

$$\mathbb{I}^{u,d} := \{\mathscr{S} \,|\, S_{j+1} \in [dSj, uS_j], \text{ for } j = 0,1,2,\dots\}.$$

Basically, this means that we assume that future asset prices will lie in between an interval characterized by an upper bound uS_j and a lower bound dS_j. This, opposed to the binomial framework, where the next asset price can assume only one of the two extreme values of the interval.
A *model* $\mathbb{M}$ of possible realizations of stock prices is a collection of all potential price paths, i.e., a sequence of real numbers

$$\mathbb{M} \subset (\mathbb{R}^+)^{N+1}.$$

Next, we introduce the definition of a strategy.

Definition 1 *A strategy is a collection of strategy functions*

$$\{g_0(S_0), g_1(S_0,S_1), \dots, g_{N-1}(S_0,\dots,S_{N-1})\}$$

that at each time t_j determine the quantity of the underlying asset to be held.

To calculate in Section 6 the fair price interval for the options (see Definition 4, below), we will just consider so called *path-independent strategies*. Path-independent strategies only take into account the current price of the asset to determine the quantity to be hold. The next definition formalizes the total costs of hedging and closure that occur using some specific strategy.

Definition 2 *For a given hedging strategy $g := \{g_0(S_0), g_1(S_0,S_1), \dots, g_{N-1}(S_0,\dots,S_{N-1})\}$ and a given asset price path $\mathscr{S} := \{S_0,\dots,S_N\}$, the total cost of hedging and closure are:*

$$Q^g(F,\mathscr{S}) := F(S_N) - \textstyle\sum_{j=0}^{N-1} g_j(S_0,\dots,S_j)(S_{j+1} - S_j).$$

Basically, the costs are nothing more than the value of the option that the seller is obliged to pay to the buyer at the expiration time T minus the trading gains from hedging the asset. The next definition is that of the *cost range*.

Definition 3 *For a given model $\mathbb{M}$ and a given initial price S of the underlying, the cost range of a strategy is defined as the set of all possible total cost paths in the model that start at the given initial price:*

$$I^g(\mathbb{M},F,S) := \{Q^g(F,\mathscr{S}) \,|\, \mathscr{S} = (S_0,\dots,S_N) \in \mathbb{M},\ S_0 = S\}.$$

Next, we state the definition of a *fair price*.

Definition 4 *A price f for a European derivative with payoff function F is said to be a fair price within the model $\mathbb{M}$ if for all strategies g there are paths $\mathscr{S}_1$ and $\mathscr{S}_2$ in $\mathbb{M}$ such that*

$$Q^g(F,\mathscr{S}_1) \le f \le Q^g(F,\mathscr{S}_2).$$

This definition of a fair price may also be expressed in a different form. For any given subset I of $\mathbb{R}$, let co$I$ denote the smallest convex subset of $\mathbb{R}$ containing I. Then

$$f \in \cap_g \text{co} I^g(\mathbb{M},F,S).$$

This intersection is what we will call the *fair price interval*, denoted by FPI($\mathbb{M}$,F,S), or, shorthand FPI. This intersection can be as small as a single point. The following proposition states that, under some assumptions, the FPI will be an interval. An interval consists of a convex set in $\mathbb{R}$ that may be closed, open, half open or may consist of a single point.

Proposition 1 *Consider an interval model $\mathbb{I}^{u,d}$. For any strategy g with respect to a European derivative with continuous payoff function $F(\cdot)$ and for any initial price S_0, the cost range $I^g(\mathbb{I}^{u,d},F,S_0)$ is an interval. If, moreover, the strategy g is continuous, then this cost range interval is closed.*

A direct consequence of this proposition is stated in the following corollary.

Corollary 1 *Consider a European derivative with continuous payoff function F. Then the fair price interval FPI($\mathbb{I}^{u,d}, F, S_0$) is an interval.*

In Section 2 we stated that we are interested in two particular strategies. The first being the *extreme delta hedging* strategy. The second strategy being such that it belongs to the class of *subgradient strategies*. The definition of a subgradient strategy will be given after the definition of the subdifferential and the subgradient, below.

Definition 5 *For a given function F: $\mathbb{R}^n \to \mathbb{R}$, the subdifferential $\partial F(x)$ of F at $x \in \mathbb{R}^n$ is defined as the set of all vectors $\gamma \in \mathbb{R}^n$ such that F(y) $\geq$ F(x) + $\gamma^T (y-x)$ for all y. The elements of $\partial F(x)$ are called in this case the subgradients of F at x.*

The subdifferential is always a nonempty convex compact set.

Definition 6 *A subgradient strategy for a European derivative with convex payoff F is any strategy g such that $g(s_j) \in \partial F(S_j)$.*

An example of a subgradient strategy for a European call option is the stop loss strategy, as defined in Section 2. The reason why we are interested in the extreme delta and the subgradient strategies is given in Theorem 2, below. It is important to note that, in this theorem, we assume that we hold a short position in the derivative. The strategies that we will use are the ones that minimize worst-case costs and strategies that maximize best-case costs, respectively. The latter may be interpreted as the opposite of a strategy that maximizes worst-case gains for a trader holding a long position in the asset.

Theorem 2 *Consider a frictionless market in which the price paths of an underlying asset follow an interval model with parameters u and d, where $d < 1 < u$; the initial value S_0 of the underlying is given. Let F(·) be the payoff function of a European derivative, and assume that F is convex. We consider portfolios that consists of (1) a given short position in the option and (2) a position in the underlying asset that is determined at each time point by a trading strategy.*

1. *Lowest worst-case costs are generated by the extreme delta-hedging strategy. The corresponding costs, which we denote by f_{max}, are given by the Cox-Ross-Rubinstein price of the derivative in the binomial tree model with the same parameters as the interval model. Worst-case costs are achieved for paths in this tree model.*
2. *Highest best-case costs are generated by any subgradient strategy. The corresponding costs are equal to $f_{min} := F(S_0)$ and are realized along the constant path.*
3. *the FPI for the derivative is $[f_{min}, f_{max}]$.*

In the case of a European call option, the stop-loss strategy is best in worst-case sense for a trader holding a long position. If we want to put this last theorem in a more game theoretical aspect, then the explanation will be as follows. Recall from Definition 2 that the total costs are nothing more than the value of the call option at the time to maturity minus the trading gains. If we compute the total costs using strategies that generate lowest worst-case and highest best-case costs, then these are nothing but applying min max and max min principles, often used in game theory. With this knowledge, we can also say that:

$$Q^{lwc}(F,\mathscr{S}) := \min_g \max_S F(S_N) - \sum_{j=0}^{N-1} g_j(S_0,\ldots,S_j)(S_{j+1}-S_j)$$
$$Q^{hbc}(F,\mathscr{S}) := \max_g \min_S F(S_N) - \sum_{j=0}^{N-1} g_j(S_0,\ldots,S_j)(S_{j+1}-S_j)$$

where Q^{lwc} are the lowest worst-case costs and Q^{hbc} are the highest base case costs respectively.

FPI Algorithm For Path-Independent Strategies

In this subsection we present an algorithm to calculate the FPI for a European call option. In the execution of the algorithm, we will consider the delta hedging strategy and the stop loss strategy to determine the FPI. The algorithm

for the FPI uses the principles of dynamic programming. Let θ_j denote a state variable at time t_j that summarizes all information over the strict past $t_0,\ldots,t_{j-1}$ that is relevant to a given strategy g. We obtain the following system:

$$\begin{aligned}\theta_{j+1} &= f_j(\theta_j, S_j),\ \theta_0 \text{ fixed},\\ \gamma_j &= g_j(\theta_j, S_j).\end{aligned} \tag{4}$$

Where f_j is a state evolution function and γ_j is the hedge position at t_j according to strategy g. We can now formulate the algorithm:

Algorithm 3

Data: *Initial asset price S_0, an interval model for assets $\mathbb{I}^{u,d}$, a continuous payoff function $F(S_N)$, and a strategy g in state space form (4).*

Step 1: Initialization
Define for $S_N \in [d^N S_0, u^N S_0]$, and arbitrary values of θ_N,

$$V^{min}(N, S_N, \theta_N) = V^{max}(N, S_N, \theta_N) := F(S_N).$$

Step 2: Backward recursion
Determine for $j = N-1,\ldots,0, S_j \in [d^j S_0, u^j S_0]$, and a suitable domain for θ_j,

$$V^{min}(j, S_j, \theta_j) := \min_{v\in[d,u]} V^{min}(j+1, vS_j, f(S_j,\theta_j)) - g_j(S_j,\theta_j)(v-1)S_j$$

and

$$V^{max}(j, S_j, \theta_j) := \max_{v\in[d,u]} V^{max}(j+1, vS_j, f(S_j,\theta_j)) - g_j(S_j,\theta_j)(v-1)S_j.$$

Define $v^{min}(S_j,\theta_j)$ and $v^{max}(S_j,\theta_j)$ as respectively the minimum and maximum location for v.

Step 3: Cost interval and extreme cost paths
Define $I := [V^{min}(0, S_0, \theta_0), V^{max}(0, S_0, \theta_0)]$
and for $j = 0, 1, \ldots, N-1$,

$$S^{min}_{j+1} = v^{min}_j(S_j,\theta_j)S_j;\ S^{max}_{j+1} = v^{max}_j(S_j,\theta_j)S_j;\ S^{min}_0 = S^{max}_0 = S_0.$$

Result:
I is the cost interval $I^g(\mathbb{I}^{u,d}, F, S)$; best-case costs are attained for S^{min} and worst-case costs for S^{max}.

The implementation of this algorithm will now be discussed in more detail. At the beginning of the algorithm, several parameters have to be fixed: the number of trading moments, N, the time to maturity, T, the strike price of the European call option, X, the initial asset price, S_0, a strategy g in state space form (4), and the jump factors u and d. For computational reasons, we will assume that the jump factors u and d are such that $u = 1/d$. As strategies we will take the stop-loss strategy and the delta hedging strategy to compute the FPI, as explained in Theorem 2.
After the parameters have been chosen, we start calculating the initial values for $V^{min}(N, S_N, \theta_N)$ and $V^{max}(N, S_N, \theta_N)$ for all values in the interval $[d^N S_0, u^N S_0]$, which, at initialization is just equal to the value of the option $F(S_N)$. However, since there are infinitely many points in this interval, it is simply impossible to evaluate all points in this interval with a computer. To approximate the solution, we will generate a 'large' amount of (grid) points K in the interval $[d^N S_0, u^N S_0]$ on a uniform scale. The issue of how to choose K is discussed in Section .
In Step 2 of the algorithm, we go one step back in the tree and we again generate a number K of grid points in the interval $[d^{N-1} S_0, u^{N-1} S_0]$. Let S_{N-1} be one of those grid points. For each of these grid points, we must determine a suitable domain for θ_{N-1}. What this means is that, for each gridpoint $S_{N-1} \in [d^{N-1} S_0, u^{N-1} S_0]$, only future stock prices S_N can be obtained in the interval $[S_{N-1} * u, S_{N-1} * d]$. So only future stock prices S_N that occur in this interval are relevant for calculating the next values $V^{min}(N-1, S_N - 1, \theta_{N-1})$ and $V^{max}(N-1, S_{N-1}, \theta_{N-1})$ using the recursive

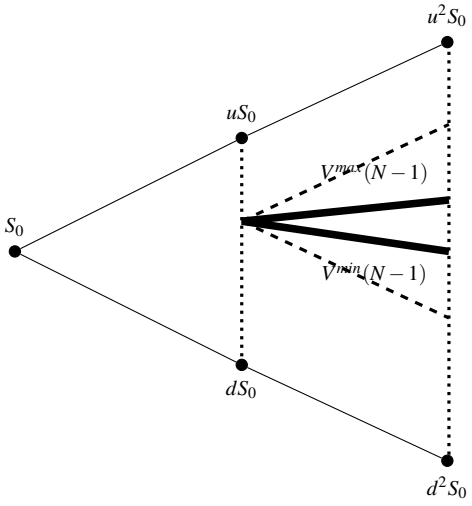


FIGURE 1. *The values of V^{min} and V^{max} are computed for S_1 for a particular strategy g at trading moment $N-1$. The dotted vertical lines indicate the grid points. The dashed lines indicate the upper and lower boundary, which indicates the maximum and minimum jump the asset is able to make. All grid points in between these two boundaries are evaluated untill V^{min} and V^{max} for this particular stock value have been found. The two thick lines depict the location of V^{min} and V^{max} for this particular value for S_1.*

relation defined in Step 2 of the algorithm, for both the stop-loss and the delta hedging strategy. We follow exactly the same procedure for all other trading moments $j = N-2,\ldots,0$, until finally we arrive at the first node of the tree. After that, we obtained two cost intervals; one for the stop-loss strategy, and one for the delta hedging strategy. The FPI is then obtained by taking the value $V^{min}(0,S_0,\theta_0)$ of the stop-loss strategy, and $V^{max}(0,S_0,\theta_0)$ of the delta hedging strategy. The idea of the algorithm is also explained for a particular S_1 in Figure 1.

The One-Step Interval Model: an Example

We will now illustrate Theorem 2 by determining the FPI for a European call option for different initial stock prices with a strike price of 100, when there is just one trading moment. (see also [5, Section 4.2.3]). First, consider the stop-loss strategy: the total costs of hedging and closure under this strategy are $Q^g = [S_1 - X]^+ - g_0(S_0)(S_1 - S_0)$, which can be expressed as

$$Q^{stop-loss} = \begin{cases} [S_1 - X]^+, & \text{if } X \geq S_0 \\ [S_1 - X]^+ - (S_1 - S_0), & \text{if } X < S_0. \end{cases}$$

The cost range of the stop-loss strategy is then given by:

$$I^{stop-loss} = \begin{cases} [uS_0 - X, 0], & \text{if } X \geq S_0 \\ [S_0 - X, (1-d)S_0], & \text{if } X < S_0. \end{cases}$$

In a similar way, we obtain the total cost of hedging and closure under the delta hedging strategy

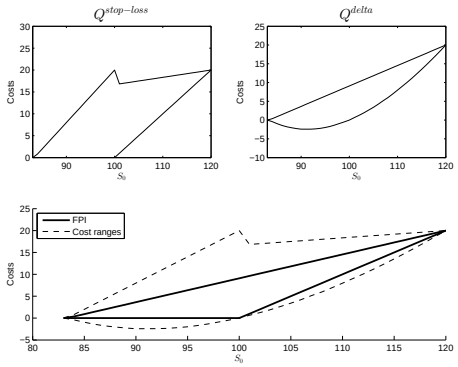
$$Q^{delta} = [S_1 - X]^+ - \frac{[uS_0 - X]^+ - [dS_0 - X]^+}{(u-d)S_0}(S_1 - S_0).$$

If we assume that X is in the interval $[dS_0, uS_0]$, it follows that $[uS_0 - X]^+ = uS_0 - X$ and $[dS_0 - X]^+ = 0$. We now have that $Q^{delta} = [S_1 - X]^+ - \frac{uS_0 - X}{(u-d)S_0}(S_1 - S_0)$. Knowing that $S_1 \in [dS_0, uS_0]$, we have that for a fixed strike price X, Q^{delta} has its maximum at $S_1 = uS_0$ and its minimum at $S_1 = X$. This means that the cost range of the delta strategy is

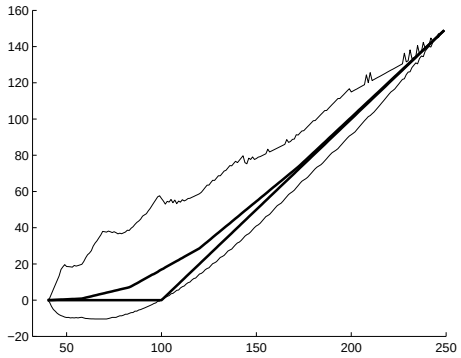
$$I^{delta} = \left[\frac{uS_0 - X)(S_0 - X)}{(u-d)S_0}, \frac{1-d}{u-d}(uS_0 - X) \right].$$

Figure 1a illustrates the various cost intervals and the fair price intervals for the stop-loss strategy and the delta hedging strategy for various levels of S_0. The lower panel merges the two upper panels. The two thick lines in this panel indicate the boundary lines of the fair price intervals. The initial stock level S_0 is chosen such that $S_0 \in [X * d^N, X * u^N]$.

Otherwise the stock price would either be too low or too high to ever cross the exercise level X, and the delta strategy would coincide with the stop-loss strategy. Figure 1b shows the obtained results using the algorithm for the same example when 5 trading moments occur.



(a) Cost range intervals and the fair price interval for a one-step interval model for $u = 1.2, d = 0.833, N = 1, X = 100$.



(b) Cost range intervals and the fair price interval for an interval model for $u = 1.2, d = 0.833, N = 5, X = 100$.

FIGURE 2. The cost range intervals and their fair price intervals.

EXTENSIONS AND REFINEMENTS OF THE NUMERICAL ALGORITHM

In this section, we will cover various extensions of the numerical algorithm to facilitate a comparison with real data. Among these extensions are the inclusion of the risk-free interest rate and transaction costs.

risk-free rate

In terms of strategies we have actually done most of the work already. The stop loss strategy with risk-free interest rate, r_j, becomes:

$$g_j(S_j) = \begin{cases} 0, & \text{if } S_j \leq X \cdot \mathrm{e}^{-r_j \cdot t_j} \\ 1, & \text{if } S_j > X \cdot \mathrm{e}^{-r_j \cdot t_j} \end{cases}$$

Whereas the corresponding delta strategy can be found in Section 2.2.

transaction costs

Usually, a trader will have to pay a certain amount to his broker for his services. A common method is to pay the broker a certain percentage of the stock price. This can be formalized as

$$w \cdot |g_j(S_j) - g_{j-1}(S_{j-1})| S_j.$$

Here w denotes the fraction that is paid to the broker. Assuming these transaction cost occur, brings on the next modification in Step 2 of the algorithm:

Step 2: Backward recursion
Determine for $j = N-1, \ldots, 0,\ S_j \in [d^j S_0, u^j S_0]$, and a suitable domain for θ_j,

$$V^{min}(j, S_j, \theta_j) := \min_{v \in [d,u]} V^{min}(j+1, vS_j, f(S_j, \theta_j) - g_j(S_j, \theta_j)(v-1)S_j) + w \cdot |g_j(S_j) - g_{j-1}(S_{j-1})| S_j$$

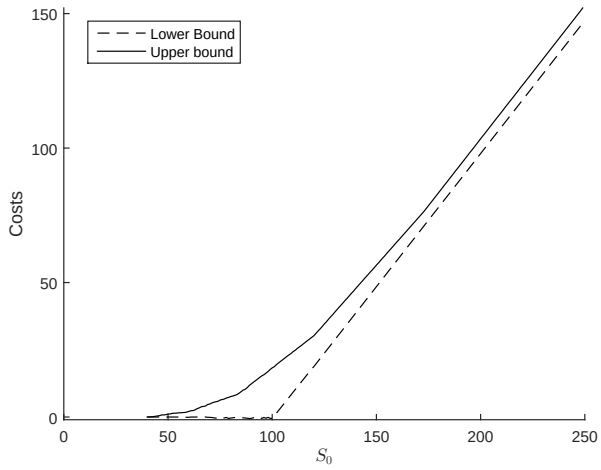


FIGURE 3. The FPI including transaction costs

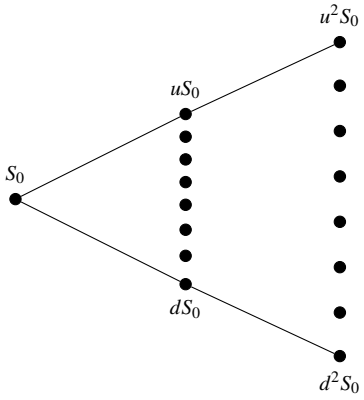
and

$$V^{max}(j,S_j,\theta_j) := \max_{v\in[d,u]} V^{max}(j+1,vS_j,f(S_j,\theta_j)-g_j(S_j,\theta_j)(v-1)S_j)+w\cdot|g_j(S_j)-g_{j-1}(S_{j-1})|S_j.$$

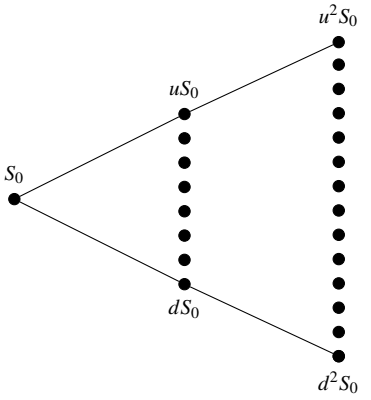
To see which impact transaction costs will have on the fair price interval we performed a small simulation study. In this study we took $T=1,\ w=0.03,\ X=100,\ \sigma=0.5$. Figure 3 shows the effect on the fair price interval of including a transaction cost fee of 3%.
The main difference with a model without transaction costs is that the lower and upper bound of the FPI do not converge any more. If the transaction fee is increased, then the divergence will occur much sooner. Another difference is that the costs are slightly higher than before, which makes sense since the transaction costs can only add to the price of a call option.

grid system

The choice of the number of grid points K is an important one. In practice, setting K to an extremely "high" value is not feasible since the number of required calculations is quadratic in NK. The algorithm would take a lot of time to produce the result. So a trade-off has to be made between accuracy and computation time by choosing K appropriately. Up until now we have taken K out of the blue, and not much has been explained regarding its choice and implementation. We will discuss two different systems that can be used. The first system is simple to implement and most commonly used. The idea is that one chooses a fixed number of grid points K, and divides them uniformly at every trading moment. The main advantage of this method is that it is relatively easy to implement. The drawback, however, is that the amount of points is at every trading moment the same, whereas the tree grows in length. For 'large' values of u,d and N, this can lead to inaccurate results. For example, if we take a drastic situation and take $u=2,d=1/2,N=30,S_0=100$ and $K=100$, then K would probably be dense enough for the interval $[dS_0,uS_0]$.



(a) Uniform method, keeping K constant



(b) Variation method, let K grow

However, for $N = 30$, the gap between $u^N S_0$ and $d^N S_0$ would be 1.0737e+11, which means we will find one point after crossing a distance of 1.0737e+9, which is not dense at all. A solution to this would be to increase K so that this distance becomes smaller, but this implies a drastic increase of calculation time as we already mentioned above. Furthermore, the increase in K could lead to an overkill for smaller gaps; setting $K = 10e+11$ might solve our initial problem, but we most likely do not need that much points for the initial interval $[dS_0, uS_0]$. So in order to tackle both of these problems, it might be a good idea to increase K as $[d^j S_0, u^j S_0]$ becomes larger for each trading moment j. The approach we will use here is as follows: we will choose an initial K, compute the distance between two points, and keep that distance the same for all gaps throughout the entire tree. This makes sure that the grid density remains the same for all trading moments and that computation time does not explode too much.

PARAMETER ESTIMATION

In this section we discuss estimation methods for the parameters used in the interval model. In particular, we discuss how to estimate the jump factors u and d, and the volatility in the market, σ. For the remainder of this section, we assume that the time passed between each trading moment is $\Delta t = \frac{T}{N}$. We make the standard assumption in the literature concerning the distribution of the return on the stocks S_j, i.e., they are lognormally distributed, i.e.,

$$ln(\frac{S_t}{S_0}) \sim N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T). \tag{5}$$

Estimating u and d

To arrive at estimates for the jump factors u and d we assume a risk-neutral framework. This implies next two things:

1. The expected return from all traded assets is the risk-free rate.
2. We can value payoffs from the derivative by calculating their expected values and discounting at the risk-free rate.

To estimate the jump factors u and d, we assume that an upward jump happens with probability p, and a downward jump happens with probability $1-p$. Since we want to match the mean and the variance of the binomial model with those of the market, we arrive at next two equations (6) and (7). Equation (6) tells us that the expected value of the stock in the binomial tree model must be equal to a stock that earns the risk-free rate. Here S denotes the asset price at the beginning of the time interval.

$$Se^{r\Delta t} = pS_u + (1-p)S_d \text{ or } e^{r\Delta t} = pu + (1-p)d. \tag{6}$$

Equation (7) provides an equality that must hold assuming variances in both models coincide:

$$pS^2u^2 + (1-p)S^2d^2 - [pSu + (1-p)Sd]^2 = S^2\sigma^2\Delta t \text{ or } pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2\Delta t. \tag{7}$$

Using (6), we find the following expression for p

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

Substituting this value for p into (7), we obtain

$$e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} = \sigma^2 \Delta t. \tag{8}$$

Equations (6) and (8) impose two conditions on p,u and d. As already mentioned before, the third condition we impose is

$$u = \frac{1}{d}. \tag{9}$$

A solution to equations (6), (8) and (9), when terms of higher order than Δt are ignored, is

$$p = \frac{a-d}{u-d};\ u = e^{\sigma\sqrt{\Delta t}};\ d = e^{-\sigma\sqrt{\Delta t}};\ \text{where } a = e^{r\Delta t}. \tag{10}$$

What is left to do now is to find an estimate for the volatility σ.

Estimating the volatility

The only parameter that is still left unspecified is the volatility σ. In empirical literature a lot of research has been done to estimate this parameter. Different approaches like, e.g., engineering and genetic programming methods are reported to estimate it (see, e.g., [8] and [1]). As this is just a first preliminary case study towards the performance of the interval model in the pricing of options, we consider three elementary, well-known, methods to arrive at an estimate. First, we consider the implied volatility as reported for the stocks by the financial market. In fact this implied volatility is based on the realizations of the option prices observed in the market. So, from that perspective, we are cheating when we use them for our simulations, as in practice they are not available. Therefore, next, we also used some simple estimation techniques to calculate the volatility based on historical data. The first technique is using the square root of the sample variance. We will call the estimate obtained using this procedure the historical volatility estimate. A second, somewhat more advanced technique, we used is the so called Garch(1,1) approach (see, e.g., [4]). Below, we will discuss both methods briefly.

Using the sample variance

A standard approach is to estimate the volatility from historical data as the square root of the sample variance. Let

$$x_i = ln(\frac{S_i}{S_{i-1}}) \text{ for } i = 1, 2, \ldots, N. \tag{11}$$

Then this estimator, s, of the standard deviation of the x_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

From equation (5), we can see that the standard deviation of u_i is given by $\sigma\sqrt{T}$. This means that s is an estimator of $\sigma\sqrt{T}$ and that σ can be estimated by $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{T}}.$$

Using the GARCH model

Another popular model to estimate volatility from historical data is the so called GARCH(1,1) model. It tries to estimate the variance by considering a weighted sum of the long run variance rate, the expected squared returns and the (estimated) variance from one period back in time. That is:

$$\sigma_n^2 = \gamma V_L + \alpha x_{n-1}^2 + \beta \sigma_{n-1}^2, \text{ with } \gamma + \alpha + \beta = 1. \tag{12}$$

Here the parameters and variables have the following meaning:

$$\begin{aligned} \sigma_{n-1}^2 &: \text{Variance of the model one period back in time.} \\ V_L &: \text{Long run variance rate.} \\ x_{n-1}^2 &: \text{Is as in } (11) \\ \gamma, \alpha, \beta &: \text{Weights that sum to unity.} \end{aligned}$$

In this approach it is assumed that the probability distribution of x_i, conditional on the variance, is normally distributed. Since the parameters of this normal distribution are unknown they have to be estimated. They can be estimated using, e.g., the maximum likelihood method. Other probability distributions then the normal distribution are also possible, but for simplicity we assumed here a normal distribution.

REAL DATA

In this section we will apply the FPI algorithm using real data from Yahoo finance [12]-[14]. We will consider several European call options and compute the FPI for different scenarios. We investigate whether the FPI gives us a good indication of the so called Bid-Ask spread. In particular we investigate which impact volatility, σ, the time to maturity, T, and the number of trading days, N, have on this fit. To make things slightly more accurate, we will bound the FPI range to be non-negative, thus excluding negative prices for call options.
To enhance a comparison with real world data, we need to calibrate the u and d parameter in our interval model. As explained in Chapter 5, we relate these parameters directly to the market volatility (see (10)) and we use three estimation techniques to calibrate it. This yields, usually, three different estimates for this parameter which we call: the implied volatility, the historical volatility and the GARCH volatility, respectively. The scenario analysis is performed for each of these three estimates.

Intel

We take historical stock data from the Intel Corporation (INTC), starting from 02-01-2014 up to 22-05-2015 [12]. The initial stock price S_0 equals 33.45 and we consider a call option that matures at 29-05-2015, meaning we have N = 5 trading days, and T = 5/252 (expressed in years). We compute the FPI for several values of X using both the historical and implied volatility estimates. The implied volatility estimate is taken from Yahoo Finance [12]. The risk-free interest, taken from [15], equals 0,1480 %. The historical volatility estimate, obtained using the stock data from the last 30 trading days, is $\sigma_{hist} = 21.54\%$. All currencies are in USD.
Table 1 shows the FPI using these two different volatility estimates. The bid and ask columns give information about the highest price a buyer is willing to pay (Bid) and the minimal price at which the seller is willing to sell (Ask).
The first thing that can be noticed is that changing the volatility has a large impact on the FPI upper bounds. The reason for the fact that the FPI is wider when the level of volatility is increased is that, since there is more uncertainty in the stock prices, stock prices can reach a wider range of values in the interval market model (see (10)). As a consequence the FPI will be larger too.
Our next observation is that the FPI tends to give higher prices for lower strike prices. This can be explained from the fact that call options are only worth something when the final stock price, S_T, exceeds the strike price, X, i.e. $S_T > X$. Clearly, if the strike price is lower, the probability that the final stock price will exceed the strike price will increase. So, the value of the option should increase.
Next we investigate if the FPI has any relation towards the Bid-Ask spread. To that end we determine the FPI for

TABLE 1. FPI bounds for several different X

X	σ_{impl}	FPI^{imp}		FPI^{hist}		B/A	
		f_{min}	f_{max}	f_{min}	f_{max}	Bid	Ask
27	159.57%	6.4500	7.1204	6.4499	6.4500	5.85	6.80
28	109.77%	5.4500	5.7517	5.4499	5.4500	5.10	5.60
29	92.97%	4.4500	4.7558	4.4499	4.4500	4.20	4.60
30	75.98%	3.4500	3.7511	3.4499	3.4500	3.10	3.60
31	49.22%	2.4500	2.6146	2.4499	2.4500	2.32	2.53
32	33.20%	1.4500	1.6061	1.4500	1.4740	1.38	1.53
33	19.53%	0.4500	0.6211	0.4500	0.6480	0.55	0.58
33.5	17.58%	0	0.3199	0	0.4011	0.24	0.25
34	17.58%	0	0.1294	0	0.1831	0.07	0.08
35	20.70%	0	0.0213	0	0.0241	0.00	0.01
36	34.77%	0	0.0391	0	0	0.00	0.02

several more Intel call options with different time to maturities and various strike prices. For all cases we used our three volatility estimates to calculate the corresponding FPI. For the estimation of the historical and GARCH volatility we used a longer estimation window for the contracts with a higher time to maturity. We used the last 150,170,200,240 and 350 days of historical stock data for them, respectively.

To determine whether the FPI could pose a decent estimate for the Bid-Ask spread we shall use the following procedure: we will compute the mean squared error (MSE) of the FPI lower bound with respect to the Bid values and the same for the FPI upper bound with respect to the Ask values. We do this under the assumption that trading occurs at a daily basis, with 252 trading days in one year, expressing T in trading years: $T = N/252$. The initial stock price S remains equal to \$33.45. The results are shown in Figures 4 through 5b.

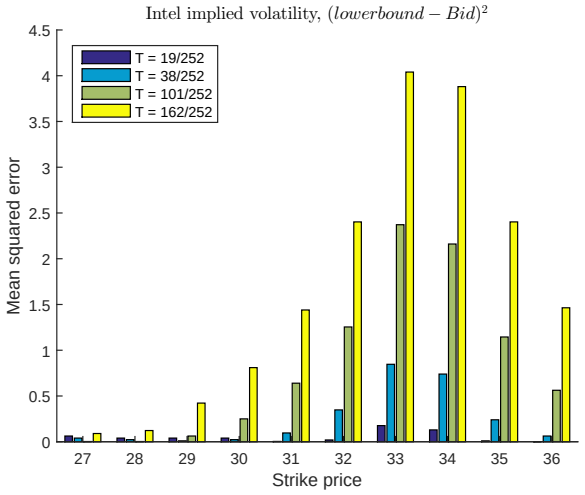


FIGURE 4. MSE w.r.t the FPI lower bound. The initial stock price S is equal to 33.45

Figure 4 displays the MSE using the implied volatility. Looking at this figure more closely a few things can be noticed. First, we see that for contracts with a longer time to maturity, the MSE is much higher.

A more interesting thing to notice is that the higher values of the MSE seem to be concentrated near the initial stock price S, whereas the MSE for strike prices further from S seem to be much lower. One possible reason for this phenomenon could be that prices for call options with a strike price near the initial S are much harder to estimate than ones with a (much) lower or higher strike price. In the latter case one has more "certainty" about whether or not a call option should be exercised or not. For instance, if we would write a call option with $X = 10$ and $S = 1000$ with a low volatility, then we would be (almost) certain that our contract will end up 'in the money', and will thus be exercised. The same reasoning can also be applied to strike prices that are higher than S. Another possible reason could be that the FPI lower bound adjusts poorly around the area where X and S are close to each other. In Figures 1a and 1b we saw that the FPI lower bound roughly consisted out of two parts: a piecewise constant part that stays around a price of zero, and a piecewise linear part that starts at $X = S$. In our model, the lower bound still tends to give a very low price

to options with a strike price near the initial stock value, whereas in reality, bidders would already pay a fair amount for such an option. It is only just after the values $S > X$ where the lower bound starts pricing call options at higher prices. This idea is illustrated in Figure 5.

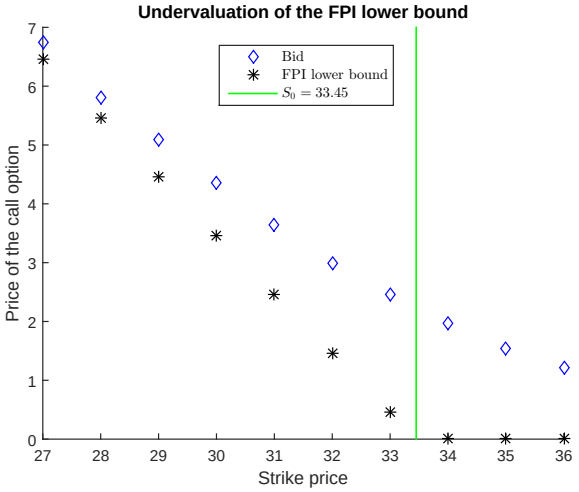
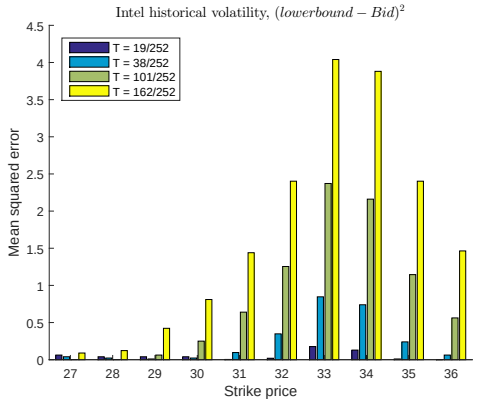
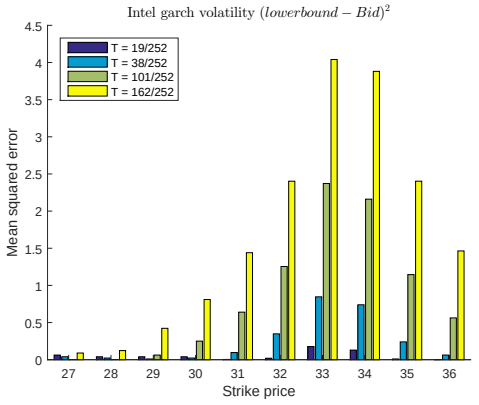


FIGURE 5. The FPI lower bound tends to give call options a lower price.

If we apply the MSE criterion again, but now using our historical and GARCH volatility estimates, respectively, then we hardly see a difference with above results. The results using these volatility estimates are shown in Figures 5a and 5b. It seems that using different volatility estimates does not have an impact on the results of the FPI lower bound at all. Recall that the lower bound is determined by the stop-loss strategy, which does not take into account the current level of volatility. We will see later that this also holds for call options having either extremely low volatilities or high volatilities.



(a) MSE w.r.t the FPI lower bound using historical volatility.



(b) MSE w.r.t the FPI lower bound using Garch(1,1) volatility.

FIGURE 6. The FPI with respect to the lower bounds.

We next consider the MSE for the FPI upper bound against the Ask values. As before, we will first consider the case when the implied volatility estimate is used. In this case, the results are much more diverse than in the cases for the lower bounds. The results using the implied volatility are shown in Figure 7.

Just like before, we see that the MSE results tend to be worse when longer times to maturities are considered. However, unlike before, we see that the results of the FPI upper bound are now far better than the ones we obtained for the FPI lower bound. Another thing that seems to be different is that the higher MSE values do not seem to be concentrated around the area where X and S are close to each other. If the historical and GARCH volatility estimates are used,

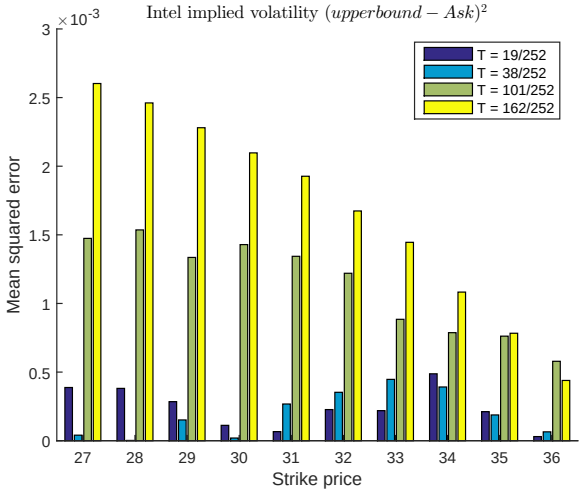
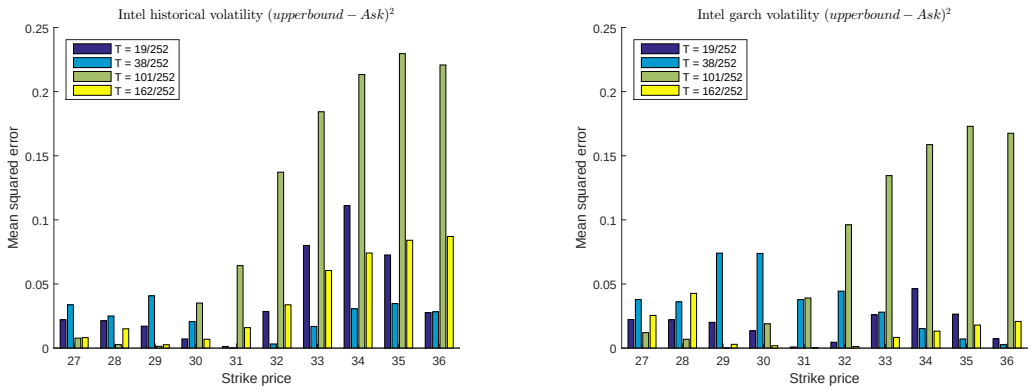


FIGURE 7. MSE w.r.t the FPI upper bound using implied volatility. The initial stock price S is equal to 33.45

results change significantly. Figures 7a and 7b illustrate this. Unlike the MSE values for the lower bounds, the MSE for the upper bounds do change when different volatility estimates are used. This is because the delta strategy, from which the upper bound is computed, depends directly on the volatility. Again the highest errors seem to occur for the long during contracts, but in this case, the longest during contract seems to perform relatively well. Also, higher MSE values are again observed near the point where $S = X$. Furthermore it seems that results based on the GARCH volatility estimate are slightly better when dealing with the areas around the initial stock price, especially for the case where $T = \frac{101}{252}$. The MSE levels between $X = 27$ and $X = 31$, however, seem to have no clear winner in terms of performance.



(a) MSE w.r.t the FPI upper bound using historical volatility.

(b) MSE w.r.t the FPI upper bound using GARCH(1,1) volatility.

FIGURE 8. The FPI with respect to the upper bounds.

Besides looking at the MSE of the upper bounds, it might also be a good idea to use the relative error (RE) as

a measure of goodness of fit. We will consider the RE only for the upper bounds [3]. Since the MSE squares all errors, we could lose some valuable information on errors. For example, we can not tell whether the FPI was too low or too high in comparison with the Ask values. For this particular reason, we will use the RE as a second tool to determine whether the upper bounds form a decent estimate for the Ask values. Figure 9 shows the involved relative errors for this study when the implied volatility estimate is used. Using the RE gives some new insights into the results. First,

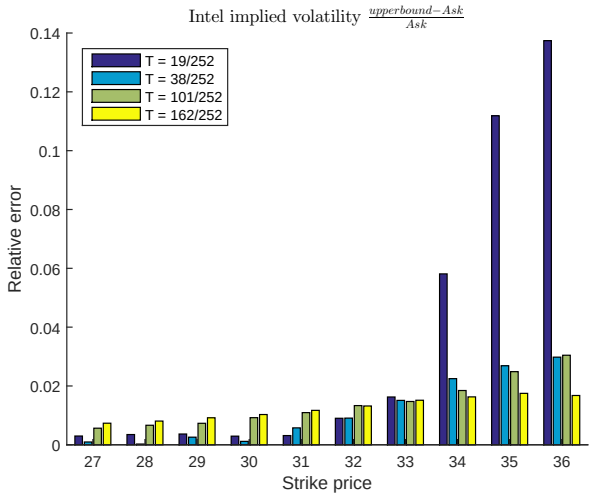
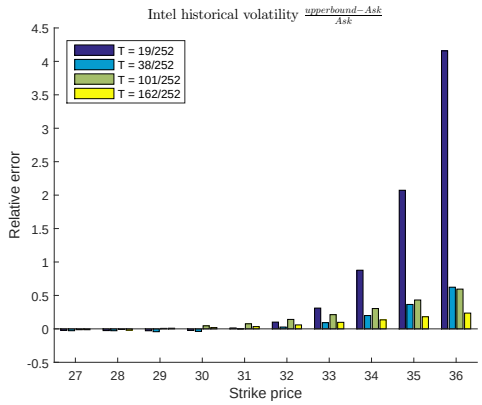


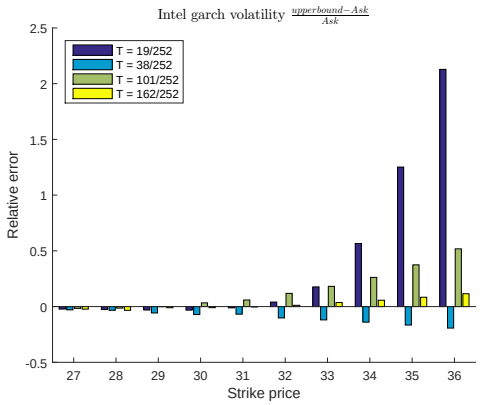
FIGURE 9. RE w.r.t the FPI upper bound using implied volatility. The initial stock price S is equal to 33.45

it seems that the lower during contracts have highest error instead of the call options with longer times to maturities, at least when looking at strike prices that are higher than the initial stock price. Also, it seems that the FPI tends to overestimate the values of the call options when using the implied volatility. This is different from what can be seen in Figures 9a and 9b. Using historical and GARCH volatility estimates, we see that the RE sometimes becomes negative, meaning that the FPI undervalued the price of the call option with respect to the Ask-values. This is especially the case for the contract with $T = \frac{38}{252}$. We also see that the errors using these volatility estimates are much worse than in the previous case, especially when call options with low values of T are considered. Results obtained using the GARCH volatility estimate outperform those obtained using the historical volatility estimate. In the remaining part of this section, we will consider several other call options characterized by lower and higher volatility trends, respectively.

[3] We do not consider the RE for the lower bounds since they can not be properly defined. The fact that many values for the lower bounds are zero, makes the RE a poor measure in this case



(a) RE w.r.t the FPI upper bound using Historical volatility.



(b) RE w.r.t the FPI upper bound using GARCH(1,1) volatility.

FIGURE 10. The FPI with respect to the upper bounds.

S&P low volatility

In this subsection we consider call options which generally have low volatility trends. In our case we will use the PowerShares S&P 500 low volatility portfolio, which we will call the S&P low for short. Data is taken from 6-01-2014 up to 29-05-2015 [14]. The initial stock price S = 37.4. We will cover several call options with 3 different times to maturity (T = 14/252, 77/252, 139/252) and two different strike prices (X = 38,39). We will again consider the MSE for the bid and ask values of the contracts with respect to the FPI lower and upper bounds, respectively. For the estimation of the historical and GARCH volatility, we used the last 170, 240 and 350 trading days from the historical stock data (starting from the contract with the lowest time to maturity to the one with the highest T). The MSE for the lower bounds of the FPI using the implied volatility estimate is shown in Figure 11. First, we observe again that the

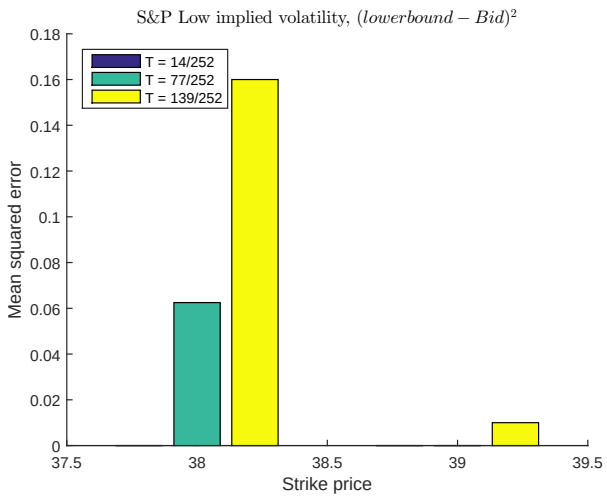


FIGURE 11. MSE w.r.t the FPI lower bound using implied volatility. The initial stock price S is equal to 37.4

MSE seems to be higher for contracts that have longer time to maturities. Also, the MSE seems to be slightly better than the ones from the Intel company that we have seen before. Most noticeably in this case is that the MSE for the

contracts with the lowest time to maturity are zero[4], as well as one for $T = \frac{77}{252}$. Since the lower bounds using historical and GARCH volatility estimates are the same as in Figure 11, their bar plots are omitted.
For the upper bounds of the FPI we neither observe much differences compared to the previous case. Results using the implied volatility estimate are shown in Figure 12.

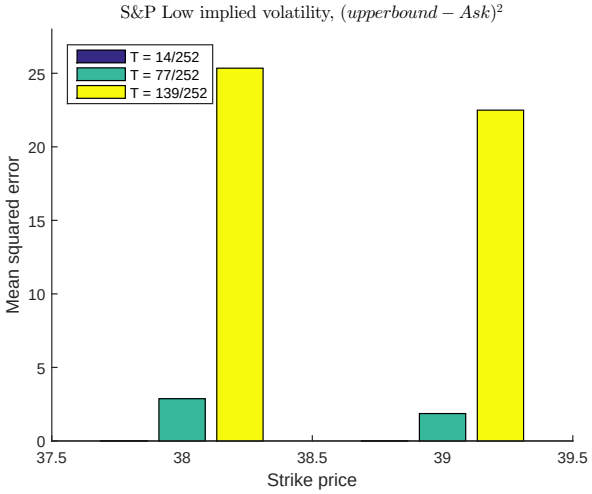
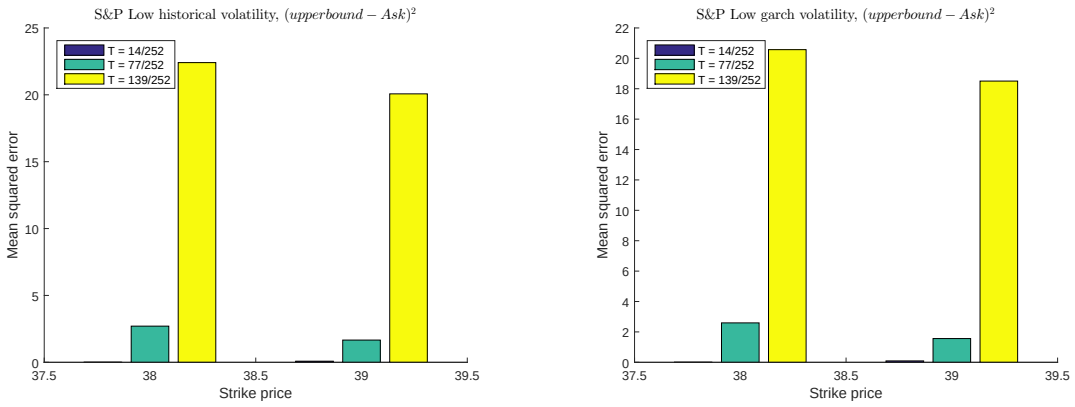


FIGURE 12. MSE w.r.t the FPI upper bound using implied volatility. The initial stock price S is equal to 37.4

The one thing that stands out is that the MSE for the longest contract is much larger than those for the other contracts. A possible explanation could be that low portfolio contracts are not as popular, since they can be exchanged by a risk-free investment earning the risk-free interest rate. In general, the amount of trading for these options is low [5]. In our case the FPI prices for this option are in the interval $[5.6, 6.3]$ whereas the Ask values are located in the interval $[0.85, 1.25]$. So, in this case, the FPI seems to price this option much higher than it should be. If we compute the MSE using the other volatility estimates, we observe only some minor changes. This, since the volatility estimates only slightly differ. Figures 12a and 12b show the results. We see that the results based on the GARCH volatility estimate slightly outperforms results obtained using the other two estimates.

[4] In fact, the MSE for this value of T is also (close to) zero in the cases where other volatility measurements are used. This explains why this value for T does not seem to be present in the bar plot.
[5] There were not many options to choose from, suggesting that traders are not actively in search for these type of portfolios



(a) MSE w.r.t the FPI upper bound using historical volatility.

(b) MSE w.r.t the FPI upper bound using GARCH(1,1) volatility.

FIGURE 13. The FPI with respect to the upper bounds.

Figures 14, 14a and 14b report the RE. These high numbers are probably due to the fact traders give a low value for these type of contracts, as already mentioned before. Furthermore, there is less need to engage in call options contracts when the stock price does not tend to fluctuate by a wide margin. Therefore it is only natural that they are valued at a lower price by traders when they are written on a stock that does not have a high volatility in general[6].

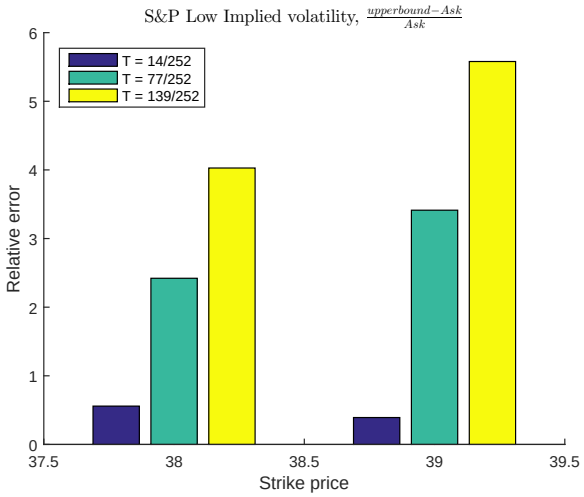
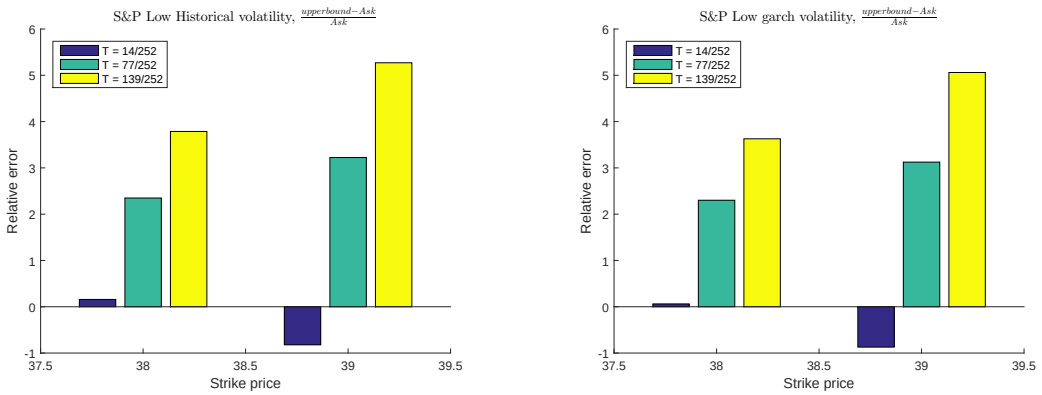


FIGURE 14. MSE w.r.t the FPI upper bound using implied volatility. The initial stock price S is equal to 37.4

[6] It might be that the FPI for this case is not relevant in this context, as it focuses on the aspect of uncertainty. It should, however, still be noted that the Bid/Ask values are still within the computed fair price interval.



(a) RE w.r.t the FPI upper bound using historical volatility.

(b) RE w.r.t the FPI upper bound using GARCH(1,1) volatility.

FIGURE 15. The FPI with respect to the upper bounds.

S&P high volatility

In this final subsection we consider call options with a very high volatility trend. For our data, we take historical stock data from the volatility S&P 500 (VIX). Our data covers dates from 2014-01-02 to 2015-05-29 [13]. We consider call options with an initial stock price of $S = 13.97$, 3 different strike prices (X ranging from 10 to 19) and different times to maturities ($T = \frac{12}{252}, \frac{36}{252}, \frac{56}{252}$) and volatility estimates using historical stock data from the last 200, 270 and 350 trading days, respectively. The results using the implied volatility estimates are shown in Figure 16.

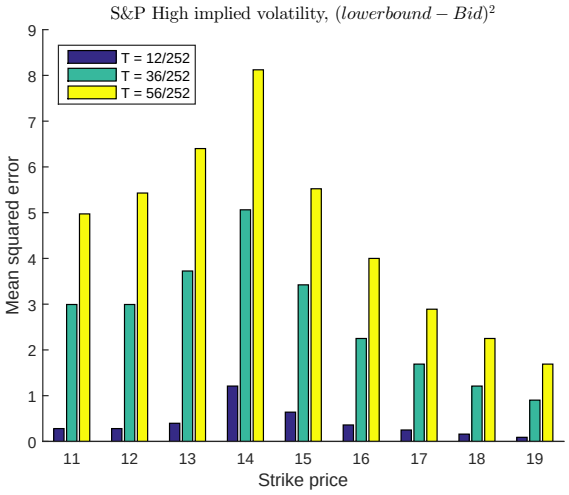


FIGURE 16. MSE w.r.t the FPI lower bound using implied volatility. The initial stock price S is equal to 13.97

Again, the contracts with longer times to maturities are exposed to higher mean squared errors. Furthermore, the lower bounds produced using the other volatility estimates coincide with these results. So they are omitted again. A possible reason why the lower bounds exhibit a high MSE could be because the volatilities are also extremely high in this case (the implied volatilities in these contracts range from 100% to 200%). However, this does not explain the relatively poor performance of the lower bound in the other cases where volatility levels were rather normal or extremely low. Another reason why the errors are high, could be because the strategy we used for the lower bounds might not be optimal for estimating the Bid values. Up until now we have seen that the lower bounds perform rather poorly when compared to the upper bounds. In the Intel case, we saw that the MSE values were rather high around $S = X$. However

in the S&P high case the errors seem to be high everywhere. In this case, bidders seem to be willing to pay a much higher price than the lower bounds predicted by the FPI (The FPI lower bound values in these contracts were almost or even zero in some cases, whereas the Bid values were much higher). Arguments for this result were already in the discussion on Figure 5, where the Bid numbers were generally higher than the values predicted by the FPI.
If we consider the MSE values for the upper bounds, we see again they perform much better than the lower bounds. The errors for the upper bounds using implied volatility are relatively low, as shown in Figure 17. Figures 17a and 17b show the MSE using the historical and GARCH volatility estimates, respectively. As before, the contracts with

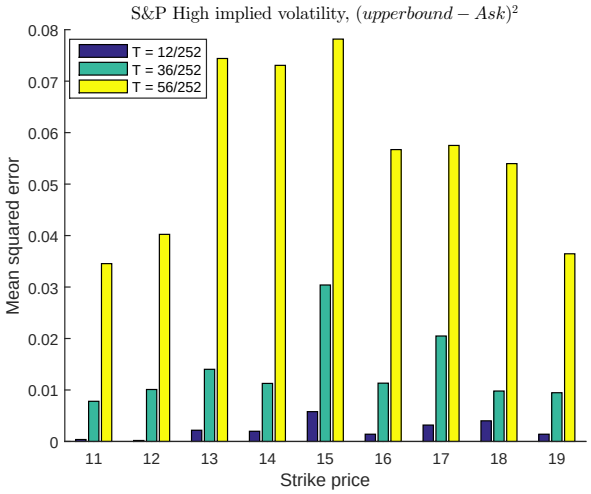
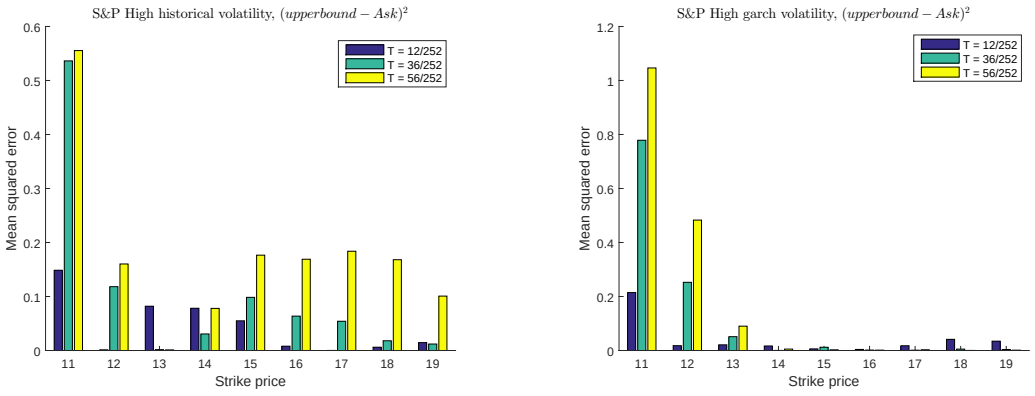


FIGURE 17. MSE w.r.t the FPI upper bound using implied volatility. The initial stock price S is equal to 13.97

the longer times to maturity have the highest MSE. Interestingly, this time the results produced using the historical volatility estimate outperform those using the Garch estimate for strike prices below the initial stock price. For strike prices above the initial stock price, the picture reverses again.



(a) MSE w.r.t the FPI upper bound using historical volatility.

(b) MSE w.r.t the FPI upper bound using GARCH(1,1) volatility.

FIGURE 18. The FPI with respect to the upper bounds.

The relative error for the upper-bound of the FPI using the implied volatility estimate are quite well, as can be seen in Figure 19. In this case there is one contract that was undervalued by the FPI, whereas in all other cases the FPI overvalued the prices of the contracts. Furthermore we observe that in most cases the performance detoriates the

higher T becomes.

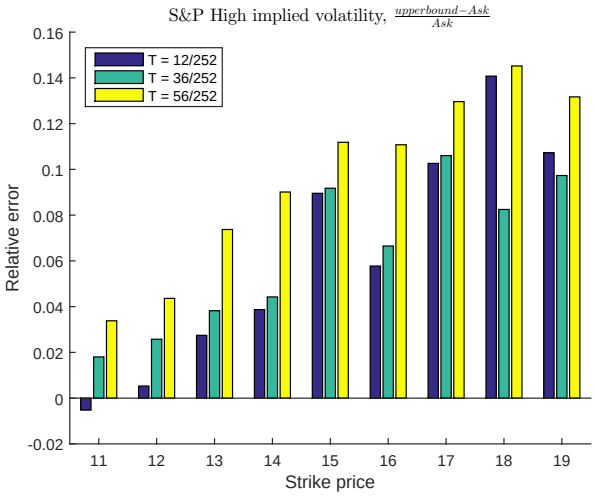
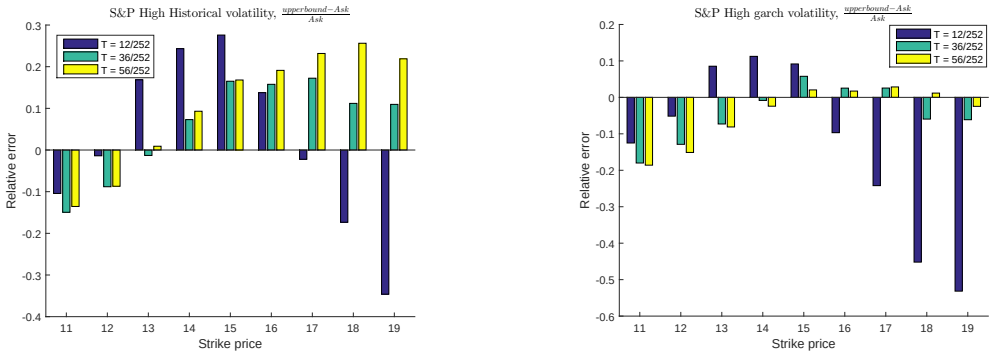


FIGURE 19. RE w.r.t the FPI upper bound using implied volatility. The initial stock price S is equal to 13.97

When using the other estimates, results change significantly. The corresponding RE plots are shown in Figures 19a and 19b. In both cases, we see that the contracts with a short time to maturity have the worst performance, switching from being overvalued to being undervalued by the FPI with respect to the Ask values. In overall it seems that the results produced using the GARCH estimates are better across most contracts. However, there are exceptions: the cases where $T = \frac{12}{252}$ and where $X > 17$. At these strike prices, the results produced using the GARCH estimate severely underestimate the price of the contract in comparison to the Ask values.



(a) RE w.r.t the FPI upper bound using Historical(1,1) volatility.

(b) RE w.r.t the FPI upper bound using GARCH(1,1) volatility.

FIGURE 20. The FPI with respect to the upper bounds.

CONCLUDING REMARKS

In this paper we discussed the interval market model as a means to model uncertainty in stock markets. This model assumes the price, S, of a stock can move to any value between two bounds uS and dS. Under this assumption we considered the question how the price of a European call option can be determined. This price is determined as an interval as well. Any price within this interval is possible. This leads to the definition of the fair price interval of the option. The bounds of this interval can be determined by determining the involved cost using two different strategies. The stop-loss and delta-hedging strategy, respectively.

In order to test whether the FPI could be a decent estimate for the so called Bid-Ask spread observed in the stock markets for options, we considered three different European call options that have low, average and high volatility trends in stock movement, respectively. We computed the mean squared error (MSE) and the relative error (RE) for the fair price interval with respect to the Bid-Ask spread. The upper bound of the FPI provides a good estimate for the Ask values, showing relatively low errors for contracts with low times to maturities, but higher MSE values for contracts with a longer timespan. For the lower bounds, the MSE values are higher by quite a margin than the ones for the upper bounds. In most cases, the prices that are computed for the lower bound are too low to accept as an estimate for the Bid-Ask spread. Regarding the tightness of the FPI, volatility has a huge impact on the upper bound, resulting in a relatively wide interval when volatility is high and a tight interval when the level of volatility is low. Furthermore, the effect on the lower bound is negligible when volatility is concerned.

The choice of what type of volatility estimation procedure to use also seemed to be an important one. In all cases, the implied volatility procedure seemed to produce the best outcomes. But it might not be the most relevant one, since it is a theoretical value and not a forecast. From the other two volatility estimation procedures we used, the historical and GARCH(1,1), the GARCH estimates seem to produce the best results in overall performance. This performance probably can be increased by using more sophisticated estimation techniques. Furthermore, the question arises whether it is not possible to estimate the u and d parameters in the interval market model more directly.

As for the estimation for the Bid values using the FPI lower bound, it might be a better idea to use a different strategy than the stop-loss in order to fit the Bid-values more closely. In overall the MSE and RE values for these estimates were quite high and did not pose a real decent fit. Probably due to its limitations in practice, the stop-loss strategy is less suited to be used as a benchmark to obtain the lower-bound of the FPI in practice. It might be worthwhile to replace it by a different subgradient strategy that can be implemented better in practice.

REFERENCES

1. Abdelmalek, W., Ben Hamida, S., Abid, F., (2009). Selecting the best forecasting-implied volatility model using genetic programming. *Journal of Applied Mathematics and Decision Sciences*, 1-19.
2. Buff, R., (2013). *Uncertain Volatility Models - Theory and Application*. Springer.
3. Dooper, S.A.M., (2015). *The Interval Market Model*. Bsc. Thesis. Department of Econometrics and Operations Research, Tilburg University, The Netherlands.
4. Enders, W., (2004). *Modelling Volatility. Applied Econometrics Time Series (Second ed.)*. Wiley.
5. Bernhard, P., Engwerda, J.C., Roorda, B., Schumacher, J.M., Kolokoltsov, V., Saint-Pierre, P., & Aubin, J.P., (2013). *The Interval Market Model in Mathematical Finance.* Birkhäuser.
6. Cox, J.C.,Ross, S.A.,& Rubinstein, M., (1979). Option pricing: a simplified approach. *Journal of Financial Economics* 7, 229-263.
7. Hull, J.C., (2012). *Options, Futures, and other Derivatives.* Pearson.
8. Ma, I., Wong, T., and Sanker, T., (2006). An engineering approach to forecast volatility of financial indices. *International Journal of Computational Intelligence* 3, 23-35.
9. Markovich, N., (2007). *Nonparametric Analysis of Univariate Heavy-Tailed Data*. Wiley.
10. Novak, S.Y., (2011). *Extreme Value Methods with Applications to Finance*. Chapman & Hall/CRC Press.
11. Quagliariello, M., (2009). *Stress-testing the Banking System: Methodologies and Applications*. Cambridge University Press.
12. `http(1)://finance.yahoo.com/q?s=INTC` (2015). Yahoo Finance, April 19, 2015.
13. `http(2)://finance.yahoo.com/q?s=^VIX`(2015) Yahoo Finance, June 2, 2015.
14. `http(3)://finance.yahoo.com/q?s=SPLV` (2015). Yahoo Finance, June 2, 2015.
15. `http(4)://nl.global-rates.com/rentestanden/libor/libor.aspx` (2015). Global rates, June 2, 2015.

Proceedings of "The 7^{th} SEAMS-UGM Conference 2015"
Topic, pp. 24–27.

On The Extension Theorem for the Symmetrized Weight Composition

Aleams Barra

Algebra Research Group, Faculty of Mathematics and Natural Sciences,
Institut Teknologi Bandung, Jln. Ganesha 10, Bandung, 40132, Indonesia
Email: barra@math.itb.ac.id

Abstract. In this paper, we give a shorter proof of the result proved by Wood in [7], that is the extension theorem holds for symmetrized weight composition on codes over Frobenius ring.

Keywords: MacWilliams Equivalence, Extension Theorem, Symmetrized Weight Composition, Frobenius Rings.

INTRODUCTION

In order to generalize the MacWilliams equivalence theorem, Goldberg [5] introduced a concept of H-coset weight W_H. He showed that any linear map between two codes over fields preserving this H-weight have to be an H-monomial map. In the case where H are the nonzero elements of the field, this result implies the Macwilliams equivalence theorem.

Later, Wood [7] introduced the notion of symmetrized weight composition with respect to a subgroup U. This concept is equivalent to the concept of the H-coset weight of Goldberg. Wood in [7] generalized the result of Goldberg for codes over Frobenius rings. By using the characterization of the Frobenius bimodule given in [6] Wood generalized the result for codes where the alphabet is a Frobenius bimodule A.

One application of the extension theorem for symmetrized weight composition is that one can reduce the problem of the extension theorem for general weights into the problem of the extension theorem for symmetrized weight composition with respect to the symmetry group of the weight. Wood used this to give a sufficient condition when a weight w satisfies the extension theorem.

In this paper we will give proof that is simpler and more direct than the one given by Wood since we are able to avoid the use of averaging character.

BASIC DEFINITIONS AND RESULTS

Let R be a finite ring with 1. A *linear code* $\mathscr{C}$ of length n over R is a left R-module of R^n. For every $x \in R^n$, the *Hamming weight* of x, denoted by $\mathrm{wt}(x)$, is the number of nonzero components of x, that is

$$\mathrm{wt}(x) := \#\{i : x_i \neq 0\}.$$

Now we are going to give some known results about characters which we will need it in our proof.

Definition 1. *Let G be an additive group. A map $\chi : G \to \mathbb{C}^\times$ is called a* character *on R if χ is a group homomorphism from G to the multiplicative group $\mathbb{C}^\times = \mathbb{C} - \{0\}$. We denoted the set of all characters on G by $\widehat{G}$ and we call this set the* character group *of G.*

We will use the following basic facts about character theory. Readers who may be interested in the proof of the statements may consult [1] or [2].

Proposition 1. *Let $G^{\mathbb{C}} := \{f \mid f : G \to \mathbb{C}\}$ the set of all functions from G to $\mathbb{C}$. The set of all characters on G is linearly independent in the complex vector space $G^{\mathbb{C}}$.*

Lemma 1. *Let $\chi_1, \ldots, \chi_s, \psi_1, \ldots, \psi_t$ be characters on G. If $\sum_{i=1}^{s} \chi_i = \sum_{j=1}^{t} \psi_j$ then $s = t$ and $\{\chi_i, \ldots, \chi_s\} = \{\psi_1, \ldots, \psi_t\}$ as a multiset.*

Frobenius Ring

Let R be a ring. We can view R as an additive group and we can consider the set of all characters $\widehat{R}$. Notice that the group of character $\widehat{R}$ has a natural (R,R)-bimodule structure. For every $r \in R$ and every $\chi \in \widehat{R}$ the left and right multiplications by r are defined by ${}^r\chi(x) := \chi(rx)$ and $\chi^r(x) := \chi(xr)$.

Definition 2. *A ring with a multiplicative identity R is called to have a* generating character, *if there exists $\chi \in \widehat{R}$ such that the map $r \mapsto {}^r\chi$ is a (left)R-module isomorphism between R and $\widehat{R}$. In this case, χ is called a* generating character *of R.*

The following results about Frobenius ring are taken from [4] and [3].

Theorem 1. *The ring R is Frobenius if and only if R has a generating character.*

Theorem 2. *Let χ be a character on R. The following are equivalent:*

1. *R has a generating character.*
2. *If I is an ideal of R and $I \subset ker\chi := \{r \in R \mid \chi(r) = 1\}$, then $I = 0$.*
3. *For each R-module V, the map from $Hom_R(V,R)$ to $\widehat{V}$ given by $g \mapsto \chi \circ g$ is an injective group homomorphism.*

SYMMETRIZED WEIGHT COMPOSITION

Let R be a finite Frobenius ring, $\mathscr{U}(R)$ the group of units in R and U is a subgroup of $\mathscr{U}(R)$. Denote by $[a_1], [a_2], \ldots, [a_t]$ all different elements of the quotient group $\mathscr{U}(R)/U$. For any $[a] = Ua \in \mathscr{U}(R)/U$ and $x = (x_1, x_2, \ldots, x_n) \in R^n$ define the *symmetrized weight composition* of x with respect to $[a]$ by

$$\mathrm{swc}_{[a]}(x) := \#\{i : x_i \in [a] = Ua\}.$$

The numerical value of $\mathrm{swc}_{[a]}(x)$ counts the number of components of x that belong to the class $[a]$. Then, for every $x \in R^n$ define a vector

$$\mathbf{swc}(x) := \left(\mathrm{swc}_{[a_1]}(x), \mathrm{swc}_{[a_2]}(x), \ldots, \mathrm{swc}_{[a_t]}(x)\right).$$

and we call it the swc_U*-vector* of x.

Example 1. *Let R be a finite field and $U = \mathscr{U}(R)$. Then $\mathscr{U}(R)/U = \{1\}$ and for every $x \in R^n$ we have that $\mathbf{swc}_U(x)$ is a vector of length one where the entry is exactly the Hamming weight of x.*

The following definitions are taken from Wood [7].

Definition 3. *Let U be a subgroup of $\mathscr{U}(R)$. A map $f : R^n \to R^n$ is called a U-monomial map if there exists $u_1, u_2, \ldots, u_n \in U$ and $\sigma \in S_n$ such that for every $x \in R^n$ we have*

$$f(x_1, x_2, \ldots, x_n) = (u_1 x_{\sigma(1)}, u_2 x_{\sigma(2)}, \ldots, u_n x_{\sigma(n)}).$$

It is not difficult to verify that a U-monomial map is linear and injective.

Definition 4. *Let $\mathscr{C}, \mathscr{C}'$ be linear codes of length n over R and U be a subgroup of $\mathscr{U}(R)$. A map $f : \mathscr{C} \to \mathscr{C}'$ is called swc_U-preserving if for every $x \in \mathscr{C}$ we have $\mathbf{swc}_U(x) = \mathbf{swc}_U(f(x))$. We say that the ring R satisfies the extension theorem for the symmetrized weight composition if every swc_U-preserving (left) linear map $f : \mathscr{C} \to \mathscr{C}'$ can be extended to a U-monomial map $\hat{f} : R^n \to R^n$.*

MAIN RESULT

Theorem 3. *Let R be a finite Frobenius ring. Then R satisfies the extension theorem for the symmetrized weight composition.*

Proof. Let $\mathscr{C}$ and $\mathscr{C}'$ be linear codes over R and let U be a subgroup of $\mathscr{U}(R)$. Suppose $f : \mathscr{C} \to \mathscr{C}'$ is an swc_U-preserving linear map. We denote the ith component of $f(x)$ by $f_i(x)$. By definition, for every $x \in \mathscr{C}$ we have $\mathbf{swc}_U(x) = \mathbf{swc}_U(f(x))$. Hence up to a left multiplication by elements in U every component of the vector $\mathbf{swc}_U(x)$ reappear as a component of $\mathbf{swc}_U(f(x))$ (and vice versa). Therefore there exists σ_x a permutation of $\{1,2,\ldots,n\}$ and $u_1,\ldots,u_n$ such that $f_i(x) = u_i x_{\sigma_x(i)}$. Now fix a $j \in \{1,2,\ldots,n\}$. Let χ be a generating character of R. Since $Uu_j = U$ then

$$\sum_{u\in U} \chi(ux_{\sigma_x(j)}) = \sum_{u\in U} \chi(uu_j x_{\sigma_x(j)}) = \sum_{u\in U} \chi(uf_j(x)).$$

Now we add up all the above equations for $j = 1,2,\ldots,n$ to get

$$\sum_{j=1}^{n}\sum_{u\in U} \chi(ux_{\sigma_x(j)}) = \sum_{j=1}^{n}\sum_{u\in U} \chi(uf_j(x)).$$

Since $\{\sigma_x(1),\ldots,\sigma_x(n)\} = \{1,2,\ldots,n\}$ then we have

$$\sum_{k=1}^{n}\sum_{u\in U} \chi(ux_k) = \sum_{j=1}^{n}\sum_{u\in U} \chi(uf_j(x)). \tag{1}$$

It is important to emphasize that equation (1) no longer depends on x.

Let $p_i : R^n \to R$ be the projection map onto the ith component, that is $p_i(x) = x_i$. By rewriting $x_k := p_k(x)$ and $f_j(x) := p_j \circ f(x)$ we can rewrite the equation (1) as

$$\sum_{k=1}^{n}\sum_{u\in U} \chi^u \circ p_k(x) = \sum_{j=1}^{n}\sum_{u\in U} \chi^u \circ p_j \circ f(x). \tag{2}$$

Since this is true for all $x \in \mathscr{C}$ we have an equation of character in the vector space $\mathscr{C}^{\mathbb{C}} = \{f \mid f : \mathscr{C} \to \mathbb{C}\}$, that is

$$\sum_{k=1}^{n}\sum_{u\in U} \chi^u \circ p_k = \sum_{j=1}^{n}\sum_{u\in U} \chi^u \circ p_j \circ f. \tag{3}$$

Take $j = 1$ and $u = 1$, we have the character ${}^1\chi \circ p_1 \circ f$ on the right hand side of the equation. By Lemma (1) we have

$$\chi \circ f_1(x) = \chi^{u_1} \circ p_{\sigma(1)}(x) = \chi \circ p_{\sigma(1)}(u_1 x)$$

for some $u_1 \in U$ and $\sigma(1) \in \{1,2,\ldots,n\}$. By the injectivity of the map $f \mapsto \chi \circ f$ (see Theorem (2)) we have $f_1(x) = u_1 x_{\sigma(1)}$.

Now

$$\sum_{u\in U} \chi^u(f_1(x)) = \sum_{u\in U} \chi^u(u_1 x_{\sigma(1)}) = \sum_{u\in U} \chi^{uu_1}(x_{\sigma(1)}) = \sum_{u\in U} \chi^u(x_{\sigma(1)}).$$

Hence

$$\sum_{u\in U} \chi^u \circ p_{\sigma(1)} = \sum_{u\in U} \chi^u \circ p_1 \circ f$$

and therefore we can decrease the size of the outersummations in the equation (3) by one to get

$$\sum_{k\neq\sigma(1)}\sum_{u\in U} \chi^u \circ p_k = \sum_{j\neq 1}\sum_{u\in U} \chi^u \circ p_j \circ f.$$

Inductively by repeating the above argument we will have $u_1,u_2,\ldots,u_n$ and a permutation σ of $\{1,2,\ldots,n\}$ such that $f_i(x) = u_i x_{\sigma(i)}$. Therefore f is a U-monomial map and can be extended to a map $\hat{f} : R^n \to R^n$. □

ACKNOWLEDGMENTS

This research is supported by Hibah Desentralisasi DIKTI 2015.

REFERENCES

1. Isaacs, I.M., *Character Theory of Finite Groups*, Courier Corporation, 2013.
2. Serre, J.P., *Linear Representations of Finite Groups*, Vol. **42**, Springer Science & Business Media, 2012.
3. Hirano, Y., On Admissible Rings, *Indagationes Mathematicae*, Vol. **8 (1)**, pages 55–59, Elsevier, 1997.
4. Honold, T., Characterization of Finite Frobenius Rings, *Archiv der Mathematik*, Vol. **76**, number 6, pages 406–415, Springer, 2001.
5. Goldberg, D.Y., A Generalized Weight for Linear Codes and a Witt-MacWilliams Theorem, *Journal of Combinatorial Theory*, Series A, 29(3), 363-367, 1980.
6. Greferath, M., Nechaev, A. and Wisbauer, R., Finite Quasi-Frobenius Modules and Linear Codes, *Journal of Algebra and its Applications*, pages 247–272. 2004.
7. Wood, J. A., Extension Theorems for Linear Codes over Finite Rings, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 329–340. Springer, 1997.

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 28-33.

Mathematics Education Students' Strategies in Solving Equations

Baiduri

Mathematics Education Department of University of Muhammadiyah Malang, East Java, Indonesia

Correspondence: Baiduri, University of Muhammadiyah Malang, East Java, Indonesia.
E-mail: baiduriumm@gmail.com

Abstract. The objective of this paper is to analyse matemathics education students'strategies in solving equations. Data were collected through responses from 167 first year students of mathematics education in University of Muhammadiyah Malang, East Java, Indonesia to the assigned tasks and the data were descriptively analysed. The results of the analysis showed that their solving of the first type of equations is still dominant perform operational procedures by transposition or working backwards. As for the second type of equations, their solving is more dominant by breaking of equations elements than to see the same elements in the equations. Then gender of students are also discussed.

Keywords: strategy, equation, solving.

INTRODUCTION

There are three main point of Algebra, those are thinking habit, variable and function as well the equality and equation [20]. It means that the subject of equations is very important for the students to be mastered.The equation is two mathematics expressions that related with the symbol 'equal to'. Many daily life problems can be presented by algebra equation, such as supply and demand of factory product. It means that the competent in solving the equation can be applied in solving problem in daily life.

Linear equations and it's solving is one of algebra components and viewed as the basic skills that mostly used in mathematics. Solving equations tend procedure rather than as the art. When it is shown, the students have done the longest and complex procedure in its solving. As the example, when the students do the equation $3(x + 1) = 6$, the steps that often taken by the students are (1) time three with the form in the bracket, (2) group the same element, and (3) divide the two function with the number (three), so it was got the $x = 1$. While the procedure that seldom done by the students are (1) divide the two side of equation with 3(three) and (2) subtract the two sides with 1 (one) , so it was got the $x = 1$. The first procedure is called conventional procedure that is often used in schools, while the the second is not conventional procedure [6]. The better way is when the students master three times how many equal 6. It this procedure is done it means the students have masterd about stucture and components of problem.

The problem solving is one of the integral activities from all of the mathematics learning [14] and he basic of the mathematics activities [17]. The ability in solving problem is very important for everyone in complex and canging society. Many phenomena in daily life can be applied in the equation. This subject is very important to be mastered by the students in solving complex problem in daily life.

There are many trategies that are developed in problem solving. Branford and Stein (cited in [2]) used IDEAL acronym to describe the process of problem solving. I-Identity the problem, D—Define and represent the problem visually, E—Explore possible strategies to solve the problem, A—Act on the chosen strategy and L—Look back and evaluate the outcomes. Krech and Novelli [10], Polya [15] and Posamentier et. al [16] have the idea that mostly mathematics problem solving consists of four steps, they are (1) understand the problem/ read the problem, 2) devise a plan/select a strategy, 3) carry out a plan/ solve the problem and 4) look back/reflect.This strategy is very common and used mostly by the people who are going to solve the problem.

Solving algebra problems (equations) different from arithmathics. In arithmetics, the result of the calculation refers to the collection of the numbers that are operated in some steps to get a certain number, and it also refers to

the answer of problem given. In other words, in algebra, the focus is in relation. The certainty of equation solution becomes the basic element. For a while, solving algebra equations has different characteristics from applying algorythme to collect the number. Algebra equations done by transforming the equations so the final result refers to expression of equations, for example $y = 7$.

Molina et.al in [13] divide two types of strategy that often done by the students in solving arithmetics problems, they are (1) making calculation to find and compare numeric value from the two element (type FC), or (2) looking the sentence and identify the certain characteristics or the relation of the elements (type LD). For a while, Hejny et.al in [4] say that this strategies that is done in solving the problem use procedural and conceptual procedure.

Kieran in [7] define many strategies that are used by students in solving linear equations, those are:

1. Using the fact number (for example: $3 + x = 5$; $5 – 3 = 2$; so $x = 2$)
2. Using calculation technique (for example: $3 + x = 9$, calculate from 3 ke 9 got $x = 6$)
3. Method *cover-up* (for example: if *replace* 9 in $2x + 9 = 5x$, become $2x + ? = 5x$, so "?" should be the same with $3x$, (that is, $3x = 9$). Replace x in $3x = 9$, result $x = 3$.)
4. Method *undoing* (*back-tracking*) (for example: we look the equations $2x + 4 = 18$ with x as the beginning, time x with 2, add with 4, and the result 18. We can cancel this operation in the contradiction. Subtrat 4 from 18, we get $2x$ (that is, $14 = 2x$). Finally divide 14 oleh 2, it was got $x = 7$.
5. Trying and error method by distributing certain numbers.
6. Tranposition method (the changing of side, the changing of symbol) (for example: in $2x + 4 = 18$, move 4 from the left to the right side by changing the symbol,the equations become $2x = 18 + (-4)$.)
7. Use the formal rule (do the same operation with the two side, balance method)

But, some researchers said that the students' mathematics knowledge is only mechanic. Started in primary schools, understanding on mathematics procedur refers to calculation ability for the four basic numeric operation that become the focus of mathematics learning. Mean while, the understanding of the conceptual mathematics is ignored [4, 5]. So that is why it is important to examine the sudents' solving problem of equations to see their mathematics understanding, especially about the equations.

METHOD

Partisipants

Participants are 167 the first semester mathematics students of University of Muhammadiyah Malang, East Java, Indonesia, who are joining in the algebra course in the 2013-2014 academic year. They consist of 41 men and 126 women. Their average age is 18.48 year and deviation standard is 0.81 year.

Instrument

The instrument of this present research consists two parts. The first part referring [13] consists of two questions. While the second part referring [18] and [5]. All these questions were used to describe mathematics education students' strategies in solving the equations.

The First Task (T1)

P1. Find the value of n so that $57 + 86 = n + 84$ is true.
P2. Find the value of d so that $345 + 576 = 342 + 574 + d$ is true.

The second task (T2)

P3. Find the solution of the equation $2(x + 5) = 6$.
P4. Find the solution of the equation $(x + \sqrt{2})^2 + 2 = 3(x + \sqrt{2})$.

Data Collection

The data focused on the response/student written answers when given a task of the equation. Data collection was conducted in two sessions before algebra course was done. Session I, students complete the first task (T1) and had maximum time is 10 minutes. Session II, students complete the second task (T2) with maximum time is 15 minutes.

Data Analysis

Data analysis was done with the coding each student's answer on each task as Knuth et. al in [8], Asquith et. al in [3] and Alibali et. al in [1]. Further to assess the reliability of the procedure coding, the coding carried out two people. The second coder was return coding 70% of the data. Agreement between codes was 98% for each reserved on each task has the same coding. Coding committed against the truth of the answers and strategies used by the students. Further data were analyzed by descriptive statistics.

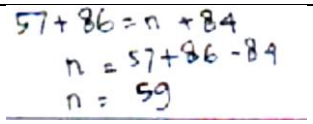
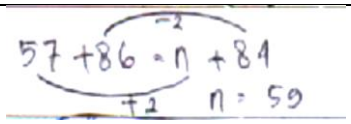
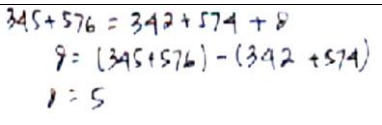
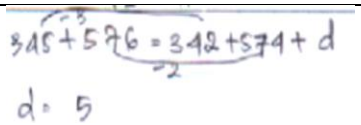
RESULTS AND DISCUSSION

Based on the students' written response toward their assignment, the coding is focused on the validity answer and strategy used to solve the problem.

Students' Problem Solving in Task I (T1)

Coding on the students' valid answer is categorized into right (**R**) and false (**F**). Meanwhile, the coding of students' strategy in solving the problem are categorized into one of three categories, those are: *transposition or do backwards* (T/WB), *pay attention to the relation of numeric value of element of both sides and make calculation* (**RC**), and only the answer (**OA**). Students' response are categorized into **T/WB,** if students do the sign-changing, symbol-changing then do the calculation operation or doing calculation operation followed with sign-changing, symbol-canging. Student's response is called **RC** response, if they focused on relationship of numerical values or entities both sides further computation. The response is categorized into **OA** if the answer only gives numeric without other information. The example of students's strategy in solving the problem can be seen in T1 in Table 1.

TABLE 1. Type of Students' Response toward T1

Problems	Student's Respon		
	T/WB	**RC**	**OA**
P1. Find the value of *n* so that 57 + 86 = *n* + 84 is true?			59
P2. Find the value of *d* so that 345 + 576 = 342 + 574 + *d* is true?			5

Based on the coding toward the validity of written answer, for **P1** male and female students who answer correctly continuously are 95.23% and 97.56 %. Meanwhile for **P2** male students who answer coreectly continuously are 93.65 % and female students are 95.12%. Although the level of correctness the two problems over 90%, it informs that type **P2** needs much pay attention in mathematic algebra teaching learning process. Dsitribution students' strategy in solving the problem assignment T1 gender based can be seen in Table 2.

TABLE 2. Distribution of Students' Strategy in Solving T1

Gender	Student's Strategies						Total
	T/WB		**RC**		**OA**		
	P1	**P2**	**P1**	**P2**	**P1**	**P2**	
Female	102	100	9	12	15	14	126
Male	29	30	4	3	8	8	41
Total	131	130	13	15	23	22	167

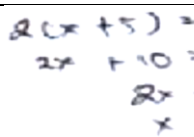
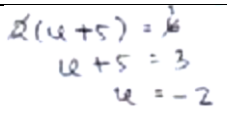
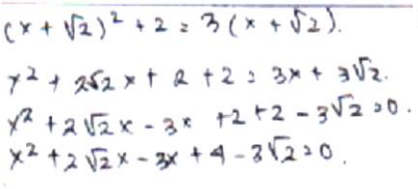
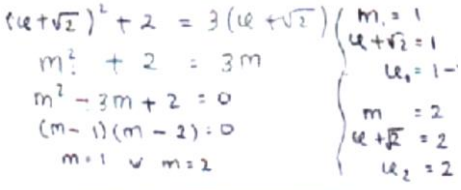
From Table 2, it can be seen that female students who used **T/WB** staretgy in solving the problem **P1**(**P2**) are 80.95% (79.37%) among all of female students or 61.08% (59.88%) among all of the students. Mean while male students with the same strategy in solving the problem **P1** (**P2**) are 70.73% (73.17%) students among male students or 17.37% (17.96) among all of the students. Female and male students who solve the problem **P1**(**P2**)

with **T/WB** startegy are 78.44% (77.84%) from all of the students. Meanwhile female students who solve equitation **P1**(**P2**) with **RC** strategy are 7.14% (9.5%) among female students or 5.39% (7.19%) from all of the students and male students who solve the problem used **RC** strategy are 9.76% (7.32%) from female students or 2.39% (1.79%) from all of the students. Female and Male students who solve the problem **P1** (**P2**) with **RC** startegy are 7.78% (7.98%) from all of the students. All of this describes that **T/WB** strategy is the most dominant strategy that cosen bay students rather than the other strategy. On the other hand, **RC** strategy is the rarely used by the students to solve the problem **P1** and **P2**. This result in line with Steinberg et.al [19] who found that there are many students know how to use transformation in solving the equation, but most of them do not take advantage of equivalent knowledge. Students' difficulties on mathematics equivalency will remain when they are in High school and even in the universities (Knuth et al, [9];. McNeil & Alibali, [12].

Students' Problem Solving in Task II (T2)

The same with T1, the coding of written answer on T2 done to the right answer that are categorized into right (**R**) or False (**F**) and the students' strategy in solving the problem are calssified into one of two categories as follows: *breaking* (**Br**) or *simplification* (**Sp**). Students'response is catgoriezd into **Br**, if student use properties of operation to break problem until they get correct unswer. Sturdents' response is categorized into **Sp** if the students used the properties of operation or substituion to simplify problem untill they get correct answer. The examples of students strategy in solving assignment T2 can be seen in Table 3.

TABLE 3. The type of Students' response towards T2

Problems	Student's Strategies	
	Br	**Sp**
P3. Find the solution of the equation $2(x + 5) = 6$.		
P4. Find the solution of the equation $(x + \sqrt{2})^2 + 2 = 3(x + \sqrt{2})$		

Based on the coding on the right written answer based on the gender, for **P3** female and male students who can answer correctly continuously are 96.83% students and 87.81 % respectively. Meanwhile for **P4** female students who can answer correctly are 0.79% and no male students. Meanwhile based on the students'strategy in responding, for **P3** are 85.03% with **Br** strategy and 9.58% with **Sp** strategy from all of the subject or 87.58% among the students with **Br** strategy and 100% from students with **Sp** strategy who can answer correctly. For **P4** only 0.59% that can answer correctly and with **Sp** strategy. Distributions of students' strategies in solving the problem assignment T2 based on the gender can be seen in Table 4.

TABLE 4. Distributions of Students' strategy in solving T2

Gender	Student's Strategies				Total
	Br		**Sp**		
	P3	**P4**	**P3**	**P4**	
Female	123	125	3	1	126
Male	38	41	3	0	41
Total	161	166	6	1	167

From Table 4, it was seen hat female students with **Br** strategy in solving **P3** (**P4**) are 96.41% (99.41%) among the total 73.65% (74.85%) the female students and 22.76% (24.55%) male students. If it is seen based on the gender, there are 97.62% (99.21%) female students among the total female students and there are 92.68% (100.00%) male students among the total male students in solving **P3** (**P4**) with **Br** strategy. Meanwhile, students with **Sp** strategy in solving problem **P3** (**P4**) are 3.59% (0.59%) among all of the female subject 1.795% (0.59%)

and male subject 1.795% (0.00%). Based on this result, strategy in breaking equations in getting the answer, convention strategy according to Johnson and Star [6] is still dominantly done by the students. Especially **P4**, most all of the students used breaking strategy in solving equation and failed in getting the right answer. It is in line with Hoch and Dreyfus' s research in [5]. The operational view in breaking equation that commonly found in elementary and yunior high school seems not to be revised or fogoten, even in many years used algebra [11].

CONCLUDING REMARK

Solving problem and mathematics learning are the two things that cannot be separated. One of the concept in learning algebra is the equation problem and its solving. So that is why referential building and learning equation problem solving in all of education level not only focussed on prosedure operation (computation), but also some strategies are needed to be developed more effectively. One of them is **RC** or **Sp** strategy. The focus of this strategy is on the relation or expression caharacteristics of both equation, so it simplifies problem and computation process.

REFERENCES

1. M.W. Alibali, E.J. Knuth, S. Hattikudur, N.M .McNeil, and A.C. Stephens, Longitudinal Examination of Middle School Students' Understanding of the Equal Sign and Equivalent Equations. *Mathematical Thinking and Learning*, **9(3)**, 221–247 (2007).
2. L.M. Arnold, A.L. Heyne, A.J. Busser, *Problem Solving: Tools And Techniques For The Park And Recreation Administrator*, (Sagamore Publishing, L.L.C. Champaign, Illinois, 2005).
3. P. Asquith, A. C. Stephens, E. J. Knuth, & M. W. Alibali, Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, **9**, 249–272 (2007).
4. M. Hejný, D. Jirotková, J. Kratochvílová, Early Conceptual Thinking. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N (Eds.). (*Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, **3**, Prague: PME, 2006), pp. 289-296.
5. M. Hoch, T. Dreyfus, Students' difficulties with applying a familiar formula in an unfamiliar context. In H. L. Chick & J. L. Vincent (Eds.), (*Proc. 29th Conf. of the Int.Group for the Psychology of Mathematics Education,* **3**, Melbourne: PME, 2005), pp. 145–152.
6. B.R. Johnson, and J.R. Star, Does Comparing Solution Methods Facilitate Conceptual and Procedural Knowledge? An Experimental Study on Learning to Solve Equations. *Journal of Educational Psychology*, **99 (3)**, 561–574 (2007).
7. C. Kieran, The learning and teaching of school algebra. In D. Grouws (Ed) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan, 1992), pp. 390-419.
8. E. Knuth, M. Alibali, S. Hattikudur, N. McNeil, & A. Stephens, The importance of equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School*, **13**, 514-519 (2008).
9. E. Knuth, A. Stephens, N. McNeil, & M. Alibali, Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, **36**, 297-312 (2006).
10. B. Krech and J. Novelli, *50 fill-in math word problems grades 4-6*, (Scholastic Inc. U.S.A, 2006).
11. N. M. McNeil, B.R. Johnson, and S. Hattikudur, Continuity in reprentation between children and adult: Arithmetic knowledge hinder undergraduates' algebraic problem solving. *Journal of Cognition and Development*, **11(4***)*, 437–457 (2010).
12. N.M. McNeil & M. Alibali, Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development*, **6(2)**, 285-306 (2005)
13. M. Molina, E. Castro and E. Castro, Third Graders´ Strategies And Use of Relational Thinking When Solving Number Sentences**. (***PME 32 and PME-NA XXX* , 2008).
14. National Council of Teachers of Mathematics (NCTM), *Principles and standards for school mathematics*. (Reston, VA: National Council of Teachers of Mathematics, 2000).
15. G. Polya, *How to Solve it.* 2nd Ed. (Princeton University Press, 1973).
16. A.S. Posamentier, D. Jaye, S. Krulik, *Exemplary Practices for Secondary Math Teachers*. (Association for Supervision and Curiculum Development. Alexandria, Virgina USA, 2007).
17. R. E. Reys, M. M. Lindquist, D. V. Lambdin, N. L. Smith, & M. N. Suydam, *Helping children learn mathematics* (6^{th} ed.). (New York: John Wiley & Sons, Inc, 2001).
18. J.R. Star and B. R.Johnson, Flexibility in problem solving: The case of equation solving, *Learning and Instruction*, **18**, 565-579 (2008).

19. Steinberg, R.; Sleeman, D.; & Ktorza, D., Algebra students' knowledge of equivalence of equations. *Journal for Research in Mathematics Education*, **22(2),** 112-121 (1990).
20. D. Foster, Making Meaning in Algebra Examining Students' Understandings and Misconceptions. *Assessing Mathematical Proficiency MSRI Publications.* **53,** 163 – 176 (2007).

Proceedings of "The 7^{th} SEAMS-UGM Conference 2015"
Topic, pp. 34–45.

Standardized Multivariate Spatial Sign Test of Independence between Two Vectors with Application on Social Economic Data

Didi Harlianto*, Yekti Widyaningsih† and Dian Lestari†

*Departement of Mathematics, Faculty of Mathematics and Natural Science, University of Indonesia, Depok, Indonesia, Email: didi.harlianto@sci.ui.ac.id
†Departement of Mathematics, Faculty of Mathematics and Natural Science, University of Indonesia, Depok, Indonesia

Abstract. Independence test of two vectors is a test that frequently used in multivariate analysis. Independence test of two vectors is used to investigate dependence of a variable in the first vector to a variable in the second vector. The common test of independency between two vectors is the likelihood ratio test. This test assumes multivariate normal of the data. This paper will introduce the new test called standardized multivariate spatial sign test which is invariant under the data transformation. The independence test is applied on variables of social economics data in some provinces of Indonesia. The variables of social economics data are divided into two variables group, that is variables group that represent health rate characteristics and variables group that represent economics characteristics. The test result based on standardized multivariate spatial sign test is health rate characteristics and economics characteristics are dependent.

Keywords: spatial sign, elliptical distribution, affine invariant, robustness

INTRODUCTION

In a research, often multiple measurement are taken on each sampling unit and then we have a data contains many variable. In many cases, the variables can be grouped based on their representation. For example, some variables may represent psychological characteristics and the other represent physical characteristics. Sometimes we want to know whether there is a relationship between the variables groups. For this objective, we need to use independence test between two vectors because each group variables can be represented by a vector. This paper will introduce a method to test independence between two vectors.

The popular independence test in the multivariate case is likelihood ratio test. Likelihood ratio test has affine invariant property and assumes multivariate normal distribution to the data. This paper introduce a new multivariate independence test that called standardized multivariate spatial sign test. This test is developed by Taskinen et all. [7]. This test has invariant property and assumes elliptically distribution to the data, so the standardized multivariate spatial sign test is more flexible to use than the ratio likelihood test. The other advantage of standardized multivariate spatial sign test, it is robust to *outlier*. In the last of this paper, the test will be implemented on social economics data from some province in Indonesia.

CONCEPTUAL THEORY

In this section, the standardized multivariate spatial sign test statistic will be developed. Figure 1 shows the standardized multivariate spatial sign testing steps in general. In step 1, we have null hypothesis

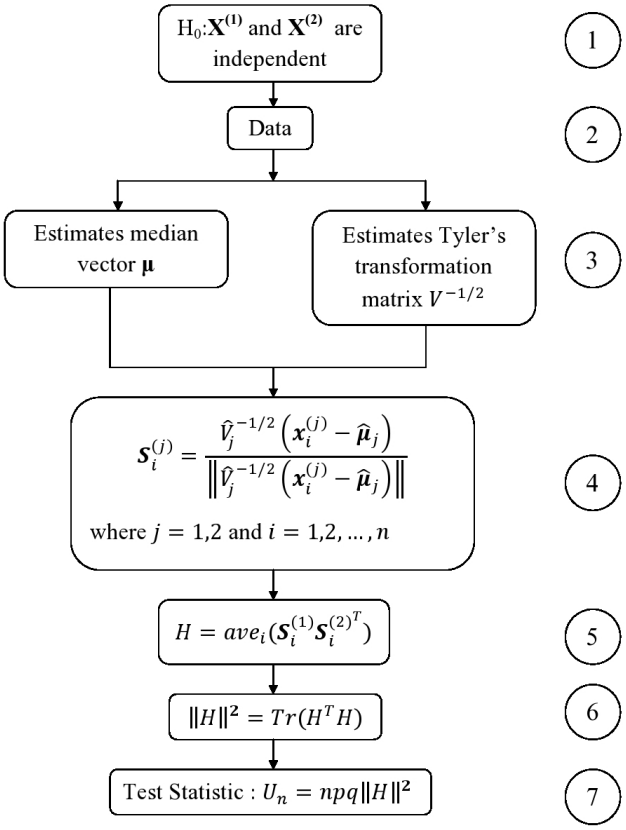


FIGURE 1. Standardized multivariate spatial sign testing steps

that $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent. The $\mathbf{X}^{(1)}$ is a vector contains the first variables group and the $\mathbf{X}^{(2)}$ contains the second variables group. Step 2 is data preparation. In step 3, median vector $\boldsymbol{\mu}$ and Tyler's transformation matrix are estimated such that the spatial sign of standardized observation

$$\mathbf{z}_i = \hat{\mathbf{V}}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}}) \tag{1}$$

satisfy

$$ave_i(\mathbf{S}(\mathbf{z}_i)) = \mathbf{0} \tag{2}$$

and

$$ave_i(\mathbf{S}(\mathbf{z}_i)\mathbf{S}^T(\mathbf{z}_i)) = \frac{1}{2}\mathbf{I}_p \tag{3}$$

where $\mathbf{S}(\mathbf{z}_i)$ is spatial sign of vector $\mathbf{z}_i$ defined as

$$\mathbf{S}(\mathbf{z}_i) = \begin{cases} \|\mathbf{z}_i\|^{-1}\mathbf{z}_i, & \text{untuk } \mathbf{z}_i \neq 0 \\ \mathbf{0}, & \text{untuk } \mathbf{z}_i = 0 \end{cases} \tag{4}$$

The estimation of median vector and Tyler's transformation matrix is done separately between the first variables group and the second variables group.

Hettmansperger and Randles [2] proposed an algorithm to compute $\hat{\boldsymbol{\mu}}$ and $\hat{V}^{-1/2}$ such that satisfy equation 2 and 3. The algorithm is shown in the Figure 2. In general, the algorithm has two main

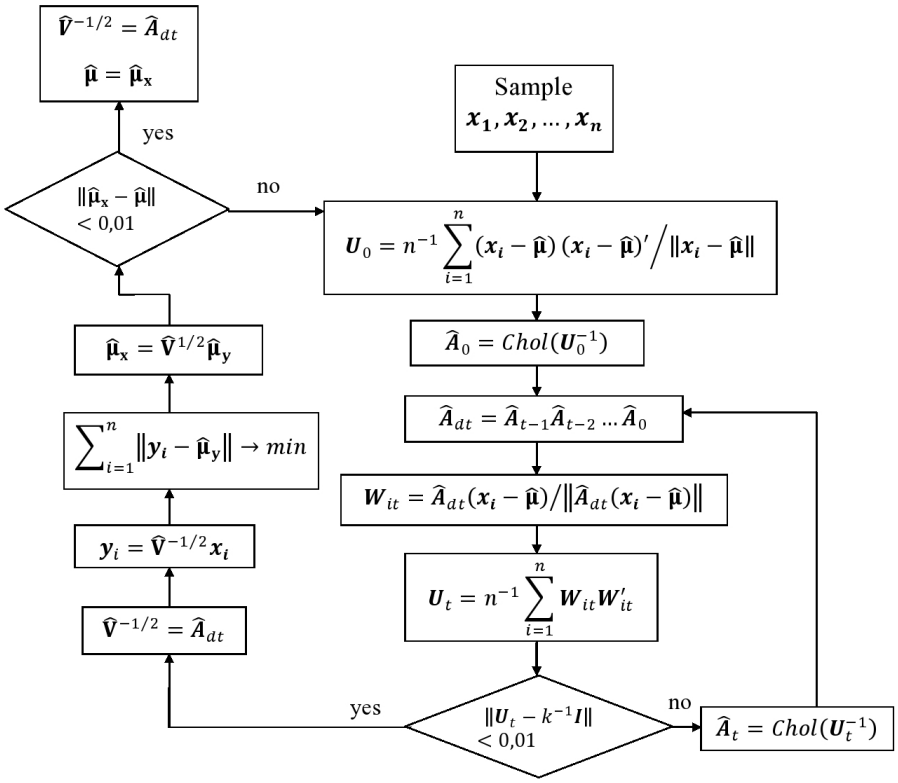


FIGURE 2. Standardized multivariate sign testing steps

procedures. The first procedure is used to find $\hat{V}^{(-1/2)}$ using a fixed value of $\boldsymbol{\mu}$. In this part, the process begins with calculation of U_0 in equation 5.

$$\mathbf{U}_0 = n^{-1} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\mu})(\mathbf{x}_i - \hat{\mu})^{\mathbf{T}} \tag{5}$$

In equation 5, the initial value of $\hat{\boldsymbol{\mu}}$ is chosen from the data range. The next, algorithm calculate $\hat{\mathbf{A}}_0 = Chol(U_0^{-1})$, where $Chol(U_0^{-1}) = \frac{1}{m_{11}} M$, M is a matrix calculated from Cholesky factorization of U_0^{-1}, and m_{11} is the element of M located at the first row and first column. In the t-th interation, algorithm calculates

$$\hat{A}_{dt} = \hat{A}_{t-1} \hat{A}_{t-2} \dots \hat{A}_0, \tag{6}$$

$$W_{it} = \hat{A}_{dt}(\mathbf{x}_i - \hat{\boldsymbol{\mu}}) / \left\| \hat{A}_{dt}(\mathbf{x}_i) - \hat{\boldsymbol{\mu}} \right\|, \tag{7}$$

and

$$U_t = n^{-1} \sum_{i=1}^{n} W_{it} W_{it}^T \tag{8}$$

If $\left\| U_t - k^{-1} I \right\|$ is small enough, where $\left\| U_t - k^{-1} I \right\|$ is sum square root of $U_t - k^{-1} I$ matrix elements, then the process stops and we get $\hat{V}^{-1/2} = \hat{A}_{dt}$. If not, the algorithm calculates $\hat{A}_{dt} = Chol(U_t^{-1})$ and the process back to equation 6, 7, and 8.

The second procedure is used to find $\hat{\boldsymbol{\mu}}$ such that satisfies equation 2 using fixed value of $\hat{V}^{-1/2}$. As the beginning of this part, each datum is transformed using $\mathbf{y}_i = \hat{V}^{-1/2}\mathbf{x}_i$. Then we find $\hat{\boldsymbol{\mu}}_{\mathbf{y}}$ that minimize $\sum_{i=1}^{n} \left\|\mathbf{y}_i - \hat{\boldsymbol{\mu}}_{\mathbf{y}}\right\|$. The median vector of the original data is found by retransformation $\boldsymbol{\mu}_{\mathbf{x}} = \hat{V}^{1/2}\hat{\boldsymbol{\mu}}_{\mathbf{y}}$. The algorithm is run until $\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}_{\mathbf{x}}$ convergent. We need the following equivariance properties in the test construction.

Lemma 1. *The standardized sign vector $\boldsymbol{S}_i$ are affine equivariant in the sense that if the $\boldsymbol{S}_i^*$ are calculated from $\boldsymbol{x}_i^* = A\boldsymbol{x}_i + \boldsymbol{b}$, $i = 1,2,\ldots,n$ with a nonsingular $k \times k$ matrix A and k-vector $\boldsymbol{b}$, then $\boldsymbol{S}_i^*$, $i = 1,2,\ldots,n$, where transformation matrix $P = \left(A\hat{V}A^T\right)^{1/2} A\hat{V}^{1/2}$ is orthogonal*

In step 4 and 5, the variance covariance matrix between marginal standardized signs and the sum square of covariance between each element of $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are defined. The standardized multivariate spatial sign test statistic is given in the next theorem

Definition 1. *The standardized multivariate spatial sign test statistic for testing the hypothesis of independence H_0 is given by*

$$U_n = npq\,\|H\|^2$$

where p and q are size of $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$, respectively.

The standardized multivariate spatial sign test statistic has affine invariant properties as stated in lemma 2.

Lemma 2. *U_n is affine invariant*

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are a sample and let $\mathbf{x}_i = \begin{pmatrix} \mathbf{x}_i^{(1)} \\ \mathbf{x}_i^{(2)} \end{pmatrix}$, where $\mathbf{x}_i^{(1)}$ and $\mathbf{x}_i^{(2)}$ are partitioned vector sizes $p \times 1$ and $q \times 1$, respectively. The test statistic T is called affine invariant if

$$\begin{aligned} T(\mathbf{x}_1^{(1)}, \mathbf{x}_2^{(1)}, \ldots, \mathbf{x}_n^{(1)}; \mathbf{x}_1^{(2)}, \mathbf{x}_2^{(2)}, \ldots, \mathbf{x}_n^{(2)}) = T(&D_1\mathbf{x}_1^{(1)} + \mathbf{b}_1, D_1\mathbf{x}_2^{(1)} + \mathbf{b}_1, \ldots, D_1\mathbf{x}_n^{(1)} + \mathbf{b}_1; \\ &D_2\mathbf{x}_1^{(2)} + \mathbf{b}_2, D_2\mathbf{x}_2^{(2)} + \mathbf{b}_2, \ldots, D_2\mathbf{x}_n^{(2)} + \mathbf{b}_2) \end{aligned}$$

for every $p \times p$ nonsingular matrix D_1, $q \times q$ nonsingular matrix D_2, p-vector $\mathbf{b}_1$, and q-vector $\mathbf{b}_2$. This property ensures that the test performance does not change if a transformation is performed to the data. The null distribution of test statistic given by theorem 1

Theorem 1. *Under H_0 and for elliptic $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$, the limiting distribution of U_n is a chi-squared distribution with pq degrees of freedom.*

APPLICATION

In this section, the standardized multivariate spatial sign test will be used to test dependence between health rate characteristics and social economics characteristics of 99 districts in three provinces in Java. They are West Java, Central Java, and East Java. The health rate characteristics are represented by two variables, namely, disease rate and infant mortality. The social economics characteristics are represented by three variables, namely, open unemployment rate, population size, poverty percentages, and regional gross domestic product. The secondary data was from BPS (Badan Pusat Statistik), 2012. If two variables group are dependent then improvement in one group will effect improvement to the other group.

The followings are standardized multivariate spatial sign test statistic value and the corresponding p-value below

$$U_n = 46.25$$

$$\text{p-value} = 2.12 \times 10^{-7}$$

As a comparison, the the likelihood ratio test result is shown below

$$W_n = 0.44$$

$$\text{p-value} = 1.89 \times 10^{-15}$$

The two test give the same conclusion, that is, the health rate characteristics and social economics characteristics of West Java, Central Java, and East Java are dependent.

To illustrate that U_n is robust against *outlier*, the p-value are recalculated while the first datum of population size 4.989939 is replaced by 49.89939. The recalculated p-values of U_n and W_n based on the data with an *outlier* are shown below.

$$\text{p-value of } U_n = 2.29 \times 10^{-7}$$

$$\text{p-value of } W_n = 4.51 \times 10^{-6}$$

It can be seen that the p-value of W_n is changed drastically than the p-value of U_n. It shows that the standardized multivariate spatial sign test is more robust against *outlier* than likelihood ratio test.

CONCLUSION

In this paper we introduce the new dependence test between two random vector. The new test is called standardized multivariate spatial sign test. The test assumes the data have elliptical distribution so this test can be used widely than ratio likelihood test. In the application section, the standardized multivariate spatial sign test has been shown more robust against *outlier* than likelihood ratio test but we need high computational computability to compute the standardized multivariate spatial sign test statistic.

ACKNOWLEDGEMENTS

The author wish to thank Department of Mathematics, University of Indonesia, for the support in this research.

APPENDIX

Proof of Theorem and Lemma

Proof of Lemma 1. First we will prove that $P = (A\hat{V}A^T)^{-1/2}A\hat{V}^{1/2}$ is a orthogonal matrix.

$$\begin{aligned} PP^T &= \left(A\hat{V}A^T\right)^{-1/2} A\hat{V}^{1/2}\hat{V}^{1/2^T}A^T \left(A\hat{V}A^T\right)^{-1/2^T} \\ &= \left(A\hat{V}A^T\right)^{-1/2} A \left(\hat{V}^{-1/2^T}\hat{V}^{-1/2}\right)^{-1} A^T \left(A\hat{V}A^T\right)^{-1/2^T} \\ &= \left(A\hat{V}A^T\right)^{-1/2} A\hat{V}A^T \left(A\hat{V}A^T\right)^{-1/2^T} \end{aligned}$$

Consider that

$$\begin{aligned} A\hat{V}A^T &= \left(\left(A\hat{V}A^T\right)^{-1/2^T}\left(A\hat{V}A^T\right)^{-1/2}\right)^{-1} \\ &= \left(A\hat{V}A^T\right)^{1/2}\left(A\hat{V}A^T\right)^{1/2^T} \end{aligned}$$

then we get

$$\begin{aligned}\left(A\hat{V}A^T\right)^{-1/2}A\hat{V}A^T\left(A\hat{V}A^T\right)^{-1/2^T} &= \left(A\hat{V}A^T\right)^{-1/2}\left(A\hat{V}A^T\right)^{1/2}\left(A\hat{V}A^T\right)^{1/2^T}\left(A\hat{V}A^T\right)^{-1/2^T}\\ &= I\end{aligned}$$

$P = (A\hat{V}A^T)^{-1/2}A\hat{V}^{1/2}$ is orthogonal matrix has proven. Next, we will prove lemma 1 using $\hat{V}^* = k\left[Tr\left(A\hat{V}A^T\right)\right]^{-1}A\hat{V}A^T$ and $\hat{\boldsymbol{\mu}}^* = A\hat{\boldsymbol{\mu}} + \mathbf{b}$.

$$\begin{aligned}\mathbf{S}_i &= S\left(\hat{V}^{*^{-1/2}}\left(\mathbf{x}_i^* - \hat{\boldsymbol{\mu}}^*\right)\right)\\ &= S\left(\left(k\left[Tr\left(A\hat{V}A\right)\right]^{-1}A\hat{V}A^T\right)^{-1/2}\left(A\mathbf{x}_i + \mathbf{b} - A\hat{\boldsymbol{\mu}} - \mathbf{b}\right)\right)\\ &= S\left(\left(k\left[Tr\left(A\hat{V}A\right)\right]^{-1}A\hat{V}A^T\right)^{-1/2}\left(A\mathbf{x}_i + \mathbf{b} - A\hat{\boldsymbol{\mu}} - \mathbf{b}\right)\right)\\ &= S\left(\left(k\left[Tr\left(A\hat{V}A\right)\right]^{-1}A\hat{V}A^T\right)^{-1/2}A\left(\mathbf{x}_i - A\hat{\boldsymbol{\mu}}\right)\right)\\ &= \frac{\left(k\left[Tr\left(A\hat{V}A\right)\right]^{-1}A\hat{V}A^T\right)^{-1/2}A\left(\mathbf{x}_i - A\hat{\boldsymbol{\mu}}\right)}{\left\|\left(k\left[Tr\left(A\hat{V}A\right)\right]^{-1}A\hat{V}A^T\right)^{-1/2}A\left(\mathbf{x}_i - A\hat{\boldsymbol{\mu}}\right)\right\|}\\ &= \frac{k\left[Tr\left(A\hat{V}A^T\right)\right]\left(A\hat{V}A^T\right)^{-1/2}A(\mathbf{x}_i - \hat{\boldsymbol{\mu}})}{k\left[Tr\left(A\hat{V}A^T\right)\right]\left\|\left(A\hat{V}A^T\right)^{-1/2}A(\mathbf{x}_i - \hat{\boldsymbol{\mu}})\right\|}\\ &= \frac{\left(A\hat{V}A^T\right)^{-1/2}A(\mathbf{x}_i - \hat{\boldsymbol{\mu}})}{\left\|\left(A\hat{V}A^T\right)^{-1/2}A(\mathbf{x}_i - \hat{\boldsymbol{\mu}})\right\|}\\ &= \frac{\left(A\hat{V}A^T\right)^{-1/2}A\hat{V}^{1/2}\hat{V}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})}{\left\|\left(A\hat{V}A^T\right)^{-1/2}A\hat{V}^{1/2}\hat{V}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})\right\|}\end{aligned}$$

because $\left(A\hat{\mathbf{V}}A^T\right)^{-1/2}A\hat{V}^{1/2}$ is orthogonal matrix then

$$\begin{aligned}\frac{\left(A\hat{V}A^T\right)^{-1/2}A\hat{V}^{1/2}\hat{V}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})}{\left\|\left(A\hat{V}A^T\right)^{-1/2}A\hat{V}^{1/2}\hat{V}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})\right\|} &= \left(A\hat{V}A^T\right)^{-1/2}A\hat{V}^{1/2}\frac{\hat{V}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})}{\left\|\hat{V}^{-1/2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})\right|}\\ &= \left(A\hat{V}A^T\right)^{-1/2}A\hat{V}^{1/2}\mathbf{S}_i\\ &= P\mathbf{S}_i\end{aligned}$$

Lemma 1 has proven.

Proof of Lemma 2. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is sample from a population. Let $\mathbf{x}_i^{(1)^*} = A_1\mathbf{x}_i^{(1)} + \mathbf{b}_1$ and $\mathbf{x}_i^{(2)^*} = A_2\mathbf{x}_i^{(2)} + \mathbf{b}_2$ where $\mathbf{x}_i^{(1)^*}$ and $\mathbf{b}_1$ is p-vector, $\mathbf{x}_i^{(2)^*}$ and $\mathbf{b}_2$ is q-vector, A_1 is nonsingular $p \times p$ matrix, A_2 is nonsingular $q \times q$ matrix. Based on lemma 1, there is orthogonal matrices P_1 and P_2 such that satisfy

$$\mathbf{S}_i^{(1)^*} = P_1 \mathbf{S}_i^{(1)}$$
$$\mathbf{S}_i^{(2)^*} = P_2 \mathbf{S}_i^{(2)}$$

Then, we get

$$\begin{aligned} H^* &= ave_i \left\{ \mathbf{S}_i^{(1)^*} \mathbf{S}_i^{(1)^{*T}} \right\} \\ &= P_1 ave_i \left\{ \mathbf{S}_i^{(1)} \mathbf{S}_i^{(2)^T} \right\} P_2^T \\ &= P_1 H P_2^T \end{aligned}$$

and we also get

$$\begin{aligned} \|H^*\|^2 &= Tr\left(\left(P_1 \mathbf{H}\, P_2^T\right)^T P_1 H P_2^T \right) \\ &= Tr\left(P_2 H^T P_1^T P_1 H P_2^T\right) \\ &= Tr\left(P_2 H^T H P_2^T\right) \\ &= Tr\left(P_2 H^T H P_2^T\right) \\ &= Tr\left(H^T H\right) \\ &= \|H\|^2 \end{aligned}$$

The consequence is $npq\|H^*\|^2 = npq\|H\|^2$. Lemma 2 has proven.

Proof of Theorem 1. The proof will be done in spherical case. First, under H_0, we will prove that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{S}_i^{(1)} \mathbf{S}_i^{(2)^T} - \frac{1}{n}\sum_{i=1}^{n} \frac{\mathbf{x}_i^{(1)} \mathbf{x}_i^{(2)^T}}{\left\|\mathbf{x}_i^{(1)}\right\| \left\|\mathbf{x}_i^{(2)}\right\|} \right) \xrightarrow{p} \mathbf{0} \tag{9}$$

Write median vector estimate and shape matrix estimate for $\mathbf{x}_i^{(1)}, i = 1,2,\ldots,n$, as

$$\boldsymbol{\mu}_1^* = \sqrt{n}\hat{\boldsymbol{\mu}}_1 \text{ dan } V_1^* = \sqrt{n}\left(\hat{V}_1 - I_p\right) \tag{10}$$

or equivalently

$$\hat{\boldsymbol{\mu}}_1 = \frac{\boldsymbol{\mu}_1^*}{\sqrt{n}} \text{ dan } \hat{V}_1 = I_p + \frac{V^*}{\sqrt{n}} \tag{11}$$

Remember that $\hat{V}_1^{1/2}\hat{V}_1^{1/2^T} = \hat{V}_1$, then implies

$$\hat{V}_1^{1/2} = I_p + \frac{V_1^*}{2\sqrt{n}} + o_p(1)$$

Because $\hat{V}_1^{\,1/2}\hat{V}_1^{-1/2} = I_p$, then

$$\hat{V}_1^{1/2} = I_p - \frac{V_1^*}{2\sqrt{n}} + o_p(1) \tag{12}$$

Equation 11 and 12 implies

$$\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)} - \hat{\boldsymbol{\mu}}_1\right) = \mathbf{x}_i^{(1)} - \frac{1}{2\sqrt{n}} V_1^* \mathbf{x}_i^{(1)} - \frac{1}{\sqrt{n}}\boldsymbol{\mu}_1^* + o_p(1) \tag{13}$$

Note that

$$\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|^2=\left(\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right)^T\left(\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right)$$

and the fact that

$$\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|=\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|^2$$

such that we get

$$\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|=\left\|\mathbf{x}_i^{(1)}\right\|-\frac{1}{2\sqrt{n}}\frac{\mathbf{x}_i^{(1)^T}V_1^*\mathbf{x}_i}{\left\|\mathbf{x}_i^{(1)}\right\|}-\frac{1}{\sqrt{n}}\frac{\mathbf{x}_i^{(1)^T}\boldsymbol{\mu}^*}{\left\|\mathbf{x}_i^{(1)}\right\|}+o_p(1) \tag{14}$$

The fact that

$$\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|^{-1}=1$$

implies

$$\left\|\hat{V}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|^{-1}=\frac{1}{\left\|\mathbf{x}_i^{(1)}\right\|}+\frac{1}{2\sqrt{n}}\frac{\mathbf{x}_i^{(1)^T}V_1^*\mathbf{x}_i}{\left\|\mathbf{x}_i^{(1)}\right\|^3}+\frac{1}{\sqrt{n}}\frac{\mathbf{x}_i^{(1)^T}\boldsymbol{\mu}^*}{\left\|\mathbf{x}_i^{(1)}\right\|^3}+o_p(1) \tag{15}$$

Multiply equation 13 and 15 so we get

$$\begin{aligned}\mathbf{S}_i^{(1)}=&\frac{\hat{\mathbf{V}}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)}{\left\|\hat{\mathbf{V}}_1^{-1/2}\left(\mathbf{x}_i^{(1)}-\hat{\boldsymbol{\mu}}_1\right)\right\|}\\ =&\frac{\mathbf{x}_i^{(1)}}{\left\|\mathbf{x}_i^{(1)}\right\|}+\frac{1}{\sqrt{n}}\frac{\mathbf{x}_i^{(1)^T}\boldsymbol{\mu}_1^*}{\left\|\mathbf{x}_i^{(1)}\right\|^2}\frac{\mathbf{x}_i^{(1)}}{\left\|\mathbf{x}_i^{(1)}\right\|}+\frac{1}{2\sqrt{n}}\frac{\mathbf{x}_i^{(1)^T}\mathbf{V}_1^*}{\left\|\mathbf{x}_i^{(1)}\right\|^2}\frac{\mathbf{x}_i^{(1)}}{\left\|\mathbf{x}_i^{(1)}\right\|}\\ &-\frac{1}{\sqrt{n}}\frac{\boldsymbol{\mu}_1^*}{\left\|\mathbf{x}_i^{(1)}\right\|}-\frac{1}{2\sqrt{n}}\frac{\mathbf{V}_1^*\mathbf{x}_i^{(1)}}{\left\|\mathbf{x}_i^{(1)}\right\|}+o_p(1)\end{aligned} \tag{16}$$

Similarly

$$\begin{aligned}\mathbf{S}_i^{(2)}=&\frac{\mathbf{x}_i^{(2)}}{\left\|\mathbf{x}_i^{(2)}\right\|}+\frac{1}{\sqrt{n}}\frac{\mathbf{x}_i^{(2)^T}\boldsymbol{\mu}_2^*}{\left\|\mathbf{x}_i^{(1)}\right\|^2}\frac{\mathbf{x}_i^{(2)}}{\left\|\mathbf{x}_i^{(2)}\right\|}+\frac{1}{2\sqrt{n}}\frac{\mathbf{x}_i^{(2)^T}V_2^*}{\left\|\mathbf{x}_i^{(2)}\right\|^2}\frac{\mathbf{x}_i^{(2)}}{\left\|\mathbf{x}_i^{(2)}\right\|}\\ &-\frac{1}{\sqrt{n}}\frac{\boldsymbol{\mu}_2^*}{\left\|\mathbf{x}_i^{(2)}\right\|}-\frac{1}{2\sqrt{n}}\frac{V_2^*\mathbf{x}_i^{(2)}}{\left\|\mathbf{x}_i^{(2)}\right\|}+o_p(1)\end{aligned} \tag{17}$$

Using equation 16 and 17 we get

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{S}_i^{(1)}\mathbf{S}_i^{(2)^T}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i^{(1)}\mathbf{x}_i^{(2)^T}}{\left\|\mathbf{x}_i^{(1)}\right\|\left\|\mathbf{x}_i^{(2)}\right\|}+o_p(1) \tag{18}$$

Note that equation 18 and 9 are aquivalent. Using central limit theorem we get

$$\sqrt{n}\,vec(H)\sim N_{pq}\left(\mathbf{0},(1/pq)I_{pq}\right)$$

Finally, note that U_n can be write as

$$U_n = npqTr(H^T H) = \sum_{i=1}^{p}\sum_{j=1}^{q}\left(\frac{\sqrt{n}h_{ij}}{1/\sqrt{pq}}\right)^2$$

and consequently $U_n \xrightarrow{d} \chi^2(pq)$. Theorem 1 has proven.

Data

The Following is secondary data of 99 districts in West Java, Central Java, and East Java. The data contains six variables, namely, Disease Rate, Infant Mortality, Open Unemployment Rate (OUR), Population Size, Gross Regional Domestic Product (GRDP), where the two former variables represents health rate characteristics and the three latter variables represents social economics characteristics. The data obtained from Badan Pusat Statistik (BPS), 2012.

TABLE 1. Health Rate Characteristics and Social Economics Characteristics Data

Number	District	Health Rate Characteristics		Social Economics Characteristics			
		Disease Rate	Infant Mortality	OUR	Population Size	Poverty	GRDP
1	Kabupaten Bogor	1.76	4295	0.0907	4.989939	8.83	9.59
2	Kabupaten Sukabumi	9.86	2693	0.0974	2.408338	9.79	21.32
3	Kabupaten Cianjur	4.81	2074	0.1126	2.231107	13.18	22.27
4	Kabupaten Bandung	5.16	2552	0.1160	3.307396	8.33	56.48
5	Kabupaten Garut	5.03	2113	0.0638	2.481152	12.72	30.15
6	Kabupaten Tasikmalaya	10.90	1203	0.0490	1.722514	11.76	15.21
7	Kabupaten Ciamis	9.48	2037	0.0528	1.562886	9.63	21.18
8	Kabupaten Kuningan	3.88	879	0.0709	1.056275	13.7	11.03
9	Kabupaten Cirebon	5.00	2129	0.1604	2.110147	14.96	22.88
10	Kabupaten Majalengka	13.60	1289	0.0671	1.189191	14.46	11.81
11	Kabupaten Sumedang	13.30	732	0.0742	1.124902	11.87	14.92
12	Kabupaten Indramayu	10.39	1429	0.0775	1.696598	15.44	23.30
13	Kabupaten Subang	3.68	1424	0.0801	1.497501	12.49	17.20
14	Kabupaten Purwakarta	6.27	495	0.0926	0.882799	9.57	19.23
15	Kabupaten Karawang	4.14	2088	0.1125	2.198978	11.11	69.41
16	Kabupaten Bekasi	2.08	1529	0.0778	2.786638	5.25	114.31
17	Kabupaten Bandung Barat	1.08	848	0.1009	1.563389	13.35	21.72
18	Kota Bogor	1.30	1521	0.0933	0.987448	8.48	17.32
19	Kota Sukabumi	8.38	697	0.1163	0.308508	8.42	6.66
20	Kota Bandung	3.75	2477	0.0917	2.461931	4.55	111.12
21	Kota Cirebon	9.27	444	0.1250	0.302772	11.1	13.22
22	Kota Bekasi	1.00	1524	0.0875	2.448291	5.56	45.86
23	Kota Depok	2.82	1222	0.0942	1.835957	2.46	20.00
24	Kota Cimahi	7.52	613	0.0857	0.560659	6.68	15.54
25	Kota Tasikmalaya	9.43	582	0.0703	0.653085	18.94	10.15
26	Kota Banjar	14.25	141	0.0620	0.18003	7.79	2.14
27	Kabupaten Cilacap	11.48	73549	0.074	1.679864	15.92	49.91
28	Kabupaten Banyumas	9.31	70999	0.0506	1.603037	19.44	12.77
29	Kabupaten Purbalingga	11.80	38496	0.0514	0.877489	21.19	7.27
30	Kabupaten Banjarnegara	18.16	39493	0.0376	0.890962	18.87	8.21
31	Kabupaten Kebumen	10.51	51261	0.0366	1.181678	22.40	7.91
32	Kabupaten Purworejo	14.95	30975	0.0328	0.708483	16.32	7.87
33	Kabupaten Wonosobo	12.98	33822	0.0537	0.771447	22.50	4.78
34	Kabupaten Magelang	6.75	52199	0.0447	1.219371	13.97	9.74
35	Kabupaten Boyolali	11.17	40846	0.0452	0.953317	13.88	9.98
36	Kabupaten Klaten	10.36	49656	0.0366	1.153047	16.71	13.53

37	Kabupaten Sukoharjo	9.69	36615	0.0598	0.848718	10.16	12.26
38	Kabupaten Wonogiri	6.58	40823	0.036	0.946373	14.67	7.93
39	Kabupaten Karanganyar	8.78	36219	0.0579	0.838762	14.07	11.47
40	Kabupaten Sragen	9.34	38211	0.06	0.875283	16.72	8.56
41	Kabupaten Grobogan	10.60	58628	0.0433	1.339127	16.14	8.05
42	Kabupaten Blora	14.69	37407	0.0488	0.847125	15.11	5.09
43	Kabupaten Rembang	16.61	26923	0.058	0.608548	21.88	5.95
44	Kabupaten Pati	10.34	53594	0.122	1.219993	13.61	11.53
45	Kabupaten Kudus	6.93	35074	0.0585	0.807005	8.63	36.96
46	Kabupaten Jepara	10.02	49795	0.042	1.144916	9.38	11.22
47	Kabupaten Demak	5.62	47938	0.0844	1.091379	16.73	7.17
48	Kabupaten Semarang	13.19	42018	0.0488	0.968383	9.40	13.85
49	Kabupaten Temanggung	14.41	32341	0.034	0.73072	12.32	6.20
50	Kabupaten Kendal	9.59	41816	0.0634	0.926325	13.17	13.43
51	Kabupaten Batang	13.14	32805	0.059	0.728578	12.40	6.49
52	Kabupaten Pekalongan	10.72	37759	0.0507	0.861366	13.86	8.93
53	Kabupaten Pemalang	10.20	55731	0.0482	1.285024	19.28	9.77
54	Kabupaten Tegal	8.11	63734	0.0605	1.421001	10.75	9.80
55	Kabupaten Brebes	14.94	76499	0.082	1.77048	21.12	18.03
56	Kota Magelang	16.49	5507	0.0871	0.120447	10.31	2.61
57	Kota Surakarta	5.33	22651	0.0607	0.509576	12.01	12.18
58	Kota Salatiga	7.14	8787	0.0669	0.17748	7.11	2.04
59	Kota Semarang	10.66	75150	0.0582	1.629924	5.13	54.38
60	Kota Pekalongan	11.15	12997	0.0744	0.290347	9.47	4.64
61	Kota Tegal	13.50	11238	0.0849	0.244632	10.04	3.08
62	Kabupaten Pacitan	12.21	23027	0.0116	0.543391	17.23	4.21
63	Kabupaten Ponorogo	15.15	36318	0.0326	0.857623	11.72	9.49
64	Kabupaten Trenggalek	7.67	28996	0.0314	0.678876	14.16	8.53
65	Kabupaten Tulungagung	8.01	44302	0.0318	1.002113	9.37	20.63
66	Kabupaten Blitar	14.19	47200	0.0286	1.126556	10.71	15.37
67	Kabupaten Kediri	10.09	66172	0.0416	1.518121	13.67	19.64
68	Kabupaten Malang	4.88	107216	0.0379	2.48712	11.01	46.83
69	Kabupaten Lumajang	9.47	43378	0.047	1.014575	12.36	17.46
70	Kabupaten Jember	11.48	101157	0.0391	2.362179	11.77	32.17
71	Kabupaten Banyuwangi	9.27	66966	0.034	1.568898	9.94	31.18
72	Kabupaten Bondowoso	17.34	32077	0.0375	0.745948	15.76	8.83
73	Kabupaten Situbondo	15.17	28564	0.0331	0.656691	14.30	3.99
74	Kabupaten Probolinggo	12.43	47233	0.0198	1.115267	22.15	18.87
75	Kabupaten Pasuruan	9.60	65832	0.0643	1.542837	11.54	20.02
76	Kabupaten Sidoarjo	10.42	85275	0.0521	2.024678	6.42	15.68
77	Kabupaten Mojokerto	10.95	44343	0.0342	1.049967	10.67	23.64
78	Kabupaten Jombang	12.15	51823	0.0669	1.21756	12.19	15.95
79	Kabupaten Nganjuk	20.35	43455	0.0422	1.025515	13.17	3.12
80	Kabupaten Madiun	8.39	28708	0.0416	0.666373	13.66	8.78
81	Kabupaten Magetan	12.00	27295	0.0386	0.621273	11.46	9.30
82	Kabupaten Ngawi	11.28	34495	0.0305	0.818871	15.94	9.16
83	Kabupaten Bojonegoro	9.82	52078	0.0351	1.218457	16.60	30.04
84	Kabupaten Tuban	9.08	47692	0.0425	1.131892	17.78	22.82
85	Kabupaten Lamongan	3.08	51220	0.0498	1.193725	16.64	15.40
86	Kabupaten Gresik	8.15	51996	0.0672	1.213449	14.30	50.19
87	Kabupaten Bangkalan	3.69	39730	0.0532	0.927433	24.62	9.47
88	Kabupaten Pamekasan	5.40	34799	0.023	0.818662	19.54	6.36
89	Kabupaten Sumenep	6.94	45520	0.0119	1.05364	21.88	13.14
90	Kota Kediri	8.62	12813	0.0785	0.273679	8.11	14.16
91	Kota Blitar	11.63	6008	0.0355	0.134554	6.73	2.56
92	Kota Malang	17.85	36978	0.0768	0.835082	5.19	43.80
93	Kota Probolinggo	10.95	9546	0.0512	0.222413	18.33	6.67
94	Kota Pasuruan	9.38	10908	0.0434	0.190045	7.88	3.31

95	Kota Mojokerto	12.58	5645	0.0732	0.012255	6.46	3.54
96	Kota Madiun	5.06	8464	0.0671	0.172421	5.35	6.42
97	Kota Surabaya	7.67	134411	0.0507	2.801409	6.23	264.34
98	Kota Batu	10.02	8530	0.0341	0.194793	4.45	4.19

REFERENCES

1. P. W. Gieser, and R. H. Randles, *Journal of the American Statistical Association* **92**, 561 - 567 (1997).
2. T. P. Hettmansperger and R. H. Randles, *Biometrika* **89**, 851-860 (2002).
3. R. V. Hogg, and A. T. Craig, "Limiting Distribution" in *Introduction to Mathematical Statistics*, Prentice Hall, New Jersey, 2004, pp. 233-253.
4. M. Kayid, Convergence of Random Variables, Chapman & Hall, London, 2000.
5. D. Paindaveine, "Elliptical Symmetry" in *Encyclopedia of Environmetrics*, John Wiley & Sons Ltd, Chichester, 2012, pp. 802-807.
6. A. C. Rencher, and W. F. Christensen, "The Multivariate Normal Distribution" in *Methods of Multivariate Analysis*, Wiley, New Jersey, 2012, pp. 91-117.
7. S. Taskinen, A. Kankainen, and H. Oja. *Statistics & Probability Letters* **62**, 9-21 (2002).

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 46-54.

A Method for Solving Fuzzy Transportation Problem of Trapezoidal Number

Dipankar De

Department of Mathematics
Maharaja Bir Bikram CollegeAgartala,Tripura-799004,India.
Email: dipankardee@yahoo.co.in

Abstract. In this paper we study fuzzy transportation problem and propose a new algorithm fuzzy Russell's method for solving transportation problem where fuzzy demand and supply are in the form of trapezoidal number. We also discuss the Robust Ranking method to solve the fuzzy transportation problem. To examine the proposed method some numerical examples are solved which show that the Russell's method is easier and take little time compare to Robust method.

Keywords: Fuzzy Transportation Problem, Trapezoidal number, Fuzzy Russell's Method, Ranking of Fuzzy number, Robust Ranking method.

INTRODUCTION

In transportation problem, the distribution of products between the origin or supply centre and destination of demand in such a manner that the total cost is minimum.

Most of the existing methods were only crisp solution for the transportation problem. But in our real life demand and unit life transportation cost are uncertain due to several factors.

To deal with the uncertain situation in Transportation Problem many Mathematicians have proposed fuzzy programming method for solving FTP. A FTP is a transportation problem in which the transportation costs, supply and demand quantities are fuzzy quantities.

The concept of fuzzy mathematical programming on a general level was first introduced by Tanaka et al. [11]. Shiang Tai Liu and Chiang Kao in [9] and Chanas and Kuchta in [2] introduced a method to solve FTP. Recently Gani and Rezak in [3] obtained a TP in which supplies and demand are trapezoidal fuzzy numbers. Pandian et al. in [7] proposed for finding optimal solution for a FTP by ranking fuzzy numbers. They study the method namely, fuzzy zero point methods, for finding a fuzzy optimal solution for a fuzzy transportation problem where all parameters are trapezoidal fuzzy numbers. Narayanamoorthy et al. in [6] discussed only the algorithm for solving FTP using Fuzzy Russell's method. The Roubast Ranking method, introduced by Yoger [12], with the help of α-cut solution has been adopted a transform FTP and solve it by the Vogel Approximation Method (VAM) but not showed its optimization.

In the beginning we recall the fundamental concepts for ready reference. Next the Roubast Ranking method is studied with numerical Example. Lastly we will introduce the procedure for fuzzy Russell's method with numerical example. We compare between these two method and also test the optimality of the solution. Finally we will end with a conclusion.

PRELIMINARIES

Definition 1. *A fuzzy number* $\mu\ \tilde{a}$ *is a fuzzy subset of the real number R with membership function* $\mu_{\tilde{a}}$ *satisfying the following conditions:*

1. $\mu_{\tilde{a}}$ *is continuous from R to the closed interval [0,1]*
2. $\mu_{\tilde{a}}$ *is strictly increasing and continuous on* $[a_1,a_2]$
3. $\mu_{\tilde{a}}$ *=1 in* $[a_2,a_3]$
4. $\mu_{\tilde{a}}$ *is strictly decreasing and continuous on* $[a_3,a_4]$

Definition 2. *A trapezoidal membership function is characterized by four parameters a, b, c, d where* $a \le b \le c \le d$, *in the equation*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & if \quad x \le a \\ h(x-a)/(b-a) & if\ a \le x \le b \\ h & if\ b \le x \le c \\ h(d-x)/(d-c) & if\ c \le x \le d \\ 0 & if\ x \ge d \end{cases}$$

A fuzzy number $\tilde{A}=(a,b,c,d)$ *is said to be a trapezoidal fuzzy number if its membership function is given by,*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & if \quad x \le a \\ (x-a)/(b-a) & if\ a \le x \le b \\ 1 & if\ b \le x \le c \\ (d-x)/(d-c) & if\ c \le x \le d \\ 0 & if\ x \ge d \end{cases}$$

Trapezoidal fuzzy number becomes a triangular fuzzy number $\tilde{A}=(a,b,d)$ *if its membership function is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & if \quad x \le a \\ (x-a)/(b-a) & if\ a \le x \le b \\ (d-x)/(d-b) & if\ c \le x \le d \\ 0 & if\ x \ge d \end{cases}$$

Remarks

1. Trapezoidal fuzzy number$\tilde{A}=(a,b,c,d)$ is to be non negative fuzzy number if $a\text{-}c \ge 0$.
2. $\tilde{A}=(a,b,c,d)$ is said to be zero trapezoidal fuzzy number if $a=0$, $b=0$, $c=0$, $d=0$.
3. Fuzzy numbers $\tilde{A}1=(a_1,b_1,c_1,d_1)$ and $\tilde{A}2=(a_2,b_2,c_2,d_2)$. Are said to be equal iff $a_1= a_2$, $b_1= b_2$, $c_1= c_2$, $d_1= d_2$.

Arithmetic Operation

The following definitions of the basic arithmetic operations on fuzzy numbers [4]:

Definition 3. *Let* $\tilde{A}1 = (a_1,b_1,c_1)$ *and* $\tilde{A}2=(a_2,b_2,c_2)$ *be two triangular fuzzy numbers, then*

(1) $\tilde{A}1 \oplus \tilde{A}2 = (a_1,b_1,c_1) \oplus (a_2,b_2,c_2) =(a_1+a_2,b_1+b_2,c_1+c_2)$

(2) $\tilde{A}1 \ominus \tilde{A}2=(a_1,b_1,c_1) \ominus (a_2,b_2,c_2) =(a_1+c_2,b_1-b_2,c_1+a_2)$

(3) $\tilde{A}1 \otimes \tilde{A}2=(a_1,b_1,c_1) \otimes (a_2,b_2,c_2) =(m_1,m_2,m_3)$ *where*

m_1= *minimum* $\{a_1a_2, a_1c_2, c_1a_2, c_1c_2\}$

$m_2 = b_1b_2$

m_3= *maximum* $\{a_1a_2, a_1c_2, c_1a_2, c_1c_2\}$

Definition 4. *Let* $\tilde{A}1=(a_1,b_1,c_1,d_1)$ *and* $\tilde{A}2 ==(a_2,b_2,c_2,d_2)$ *be two trapezoidal fuzzy numbers. Then*

(1) $\tilde{A}1 \oplus \tilde{A}2=(a_1,b_1,c_1,d_1) \oplus (a_2,b_2,c_2,d_2) = (a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)$

(2) $\tilde{A}1 \ominus \tilde{A}2=(a_1,b_1,c_1,d_1) \ominus (a_2,b_2,c_2,d_2)=(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2)$

(3) $\tilde{A}1 \otimes \tilde{A}2=(a_1,b_1,c_1,d_1) \otimes (a_2,b_2,c_2,d_2)=(m_1,m_2,m_3,m_4)$ *where*

m_1= minimum $\{a_1a_2, a_1d_2, d_1a_2, d_1d_2\}$

m_2= minimum $\{b_1b_2, b_1c_2, c_1b_2, c_1c_2\}$

m_3=maximum$\{b_1b_2, b_1c_2, c_1b_2, c_1c_2\}$

m_4=maximum $\{a_1a_2, a_1d_2, d_1a_2, d_1d_2\}$

Multiplication used by Stephen et al.

$\tilde{A}1 \otimes \tilde{A}2 =$

$$[\frac{a_1}{4}(b_1+b_2+b_3+b_4),\frac{a_2}{4}(b_1+b_2+b_3+b_4),\frac{a_3}{4}(b_1+b_2+b_3+b_4),\frac{a_4}{4}(b_1+b_2+b_3+b_4)] \text{ if } R(\tilde{a})>0$$

$\tilde{A}1 \otimes \tilde{A}2 =$

$$[\frac{a_4}{4}(b_1+b_2+b_3+b),\frac{a_3}{4}(b_1+b_2+b_3+b_4),\frac{a_2}{4}(b_1+b_2+b_3+b_4),\frac{a_1}{4}(b_1+b_2+b_3+b_4)], \text{ if } R(\tilde{a})<0.$$

Definition 5. *A ranking function R:* $F(R) \rightarrow R$ *which maps each fuzzy number into the real line, F(R) represents the set of all trapezoidal fuzzy number. If R be any ranking function*

$$R(\tilde{a}) = \frac{a_1+a_2+a_3+a_4}{4}.$$

Example 1. *Let* A_1 *and* A_2 *be two trapezoidal fuzzy numbers, where*

$A_1=(1,5,6,9)$ *and* $A_2=(2,3,5,8)$*,then*

$A_1 \oplus A_2=(1,5,6,9) \oplus (2,3,5,8)=(3,8,11,17)$

$A_1 \ominus A_2=(1,5,6,9) \ominus (2,3,5,8)=(-7,0,3,7)$

$A_1 \otimes A_2=(1,5,6,9) \otimes (2,3,5,8)=(m_1,m_2,m_3,m_4)=(2,15,30,72)$

m_1= *minimum* $\{a_1a_2, a_1d_2, d_1a_2, d_1d_2\}=min\{2,8,18,72\}=2$

m_2= *minimum* $\{b_1b_2, b_1c_2, c_1b_2, c_1c_2\}=min\{15,25,18,30\}=15$

$m_3=maximum\{b_1b_2, b_1c_2, c_1b_2, c_1c_2\}=max\{15,25,18,30\}=30$

m_4=*maximum* $\{a_1a_2, a_1d_2, d_1a_2, d_1d_2\}=max\{2,8,18,72\}= 72$

Definition 6. *The* α*- cut of fuzzy trapezoidal number* $A= (a_1,a_2,a_3,a_4)$ *is defined by*

$$A_\alpha =[(a_2-a_1)\alpha +a_1,-(a_4-a_3)\alpha +a_4), \quad \forall\, \alpha \in [0,1]$$

FUZZY TRANSPORTATION PROBLEM

Let us consider a transportation system based on fuzzy with m fuzzy origins and n fuzzy destinations. Let us further assume that the transportation cost of one unit of product from i^{th} fuzzy origin to j^{th} fuzzy destination be denoted by $\tilde{c}_{ij}$. Let $\tilde{a}_i$ and $\tilde{b}_j$ be the amount of fuzzy supply of the i^{th} origin and the amount of fuzzy demand of the j^{th} destination respectively. Let x_{ij} represent the amount transported from fuzzy origin i to fuzzy destination j.

Mathematical formulation of fuzzy transportation problem is

Minimize $Z = \sum\sum \tilde{c}_{ij}\tilde{x}_{ij}$
Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = \tilde{a}_i \quad , i=1,2,\ldots m$$

$$\sum_{i=1}^{m} x_{ij} = \tilde{b}_j \qquad j=1,2,\ldots n$$

$x_{ij} \geq 0$, $i = 1,2,\ldots m$; $j = 1,2,\ldots n$.

If the total fuzzy capacity is equal to total fuzzy demand, i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

A set of x_{ij} which satisfies the row and column restriction is called a fuzzy solution.

ROBUST RANKING METHOD

Robust Ranking Method which satisfies linearity and additive properties. The Robust Ranking index of fuzzy number a is defined by

$R(a) = \int_0^1 0.5(a_\alpha^L, a_\alpha^U)d\alpha$

Where (a_α^L, a_α^U) is the α-level cut of fuzzy number a.

For the transportation problem, fuzzy objective function

$$\text{Min } z = \sum\sum \tilde{c}_{ij} x_{ij}$$

We apply Robust Ranking method to get the minimum object value z^* from the formulation

$$R(z^*) = \text{Min}z = \sum\sum R(a_{ij})x_{ij}$$

Now we calculate R(-2,0,2,8) by applying Robust Ranking method. The membership function of transportation number (-2,0,2,8)

$$\mu(x) = \begin{cases} \dfrac{x-(-2)}{0-(-2)} & -2 < x < 0 \\ \dfrac{8-x}{8-2} & 2 < x < 8 \end{cases}$$

The α-cut of fuzzy number (-2,0,2,8) is (a_α^U, a_α^L)

$$= [(a_2-a_1)\alpha+a_1, -(a_4-a_3)\alpha+a_4]$$
$$= [2\alpha-2, 8-6\alpha]$$

Using Robust Ranking method,

$\because$ $R(a_{11}) = R(-2,0,-2,8)$

$$= \int_0^1 .5(2\alpha - 2 + 8 - 6\alpha)d\alpha$$

$$= \int_0^1 .5(6 - 4\alpha)d\alpha$$

$$= .5\,[6\alpha-2\alpha^2)_0^1 = .5(6-2) = 2$$

Proceding similarly, the Robust's ranking indeces for fuzzy costs are calculated for following example.

Example 2.

To solve the Fuzzy Transportation Problem using Robust Ranking Method:

	D1	D2	D3	D4	Supply
S1	(-2,0,2,8)	(-2,0,2,8)	(-2,0,2,8)	(-1,0,1,4)	(0,2,4,6)
S2	(4,8,12,16)	(4,7,9,12)	(2,4,6,8)	(1,3,5,7)	(2,4,9,13)
S3	(2,4,9,13)	(0,6,8,10)	(0,6,8,10)	(4,7,9,12)	(2,4,6,8)
Demand	(1,3,5,7)	(0,2,4,6)	(1,3,5,7)	(1,3,5,7)	(4,10,19,27)

Solution:

$$R(a_{21})= R(4,8,12,16)=\int_0^1 .5[(a_2-a_1)\alpha+a_1,-(a_4-a_3)\alpha+a_4]d\alpha$$

$$=\int_0^1 .5[4\alpha+4,-16-4\alpha]d\alpha$$

$$=\int_0^1 .5[20]d\alpha =10$$

$$R(4,10,19,27)= \int_0^1 .5[(a_2-a_1)\alpha+a_1,-(a_4-a_3)\alpha+a_4]d\alpha$$

$$=\int_0^1 .5[(10-4)\alpha+4,-(27-19)\alpha+27]d\alpha$$

$$=\int_0^1 .5(6\alpha+4-8\alpha+27)d\alpha=\int_0^1 .5(31-2\alpha)d\alpha=.5(31\alpha-\alpha^2)_0^1$$

$$=.5(30)=15.$$

Similarly, other Rank of costs can be calculated and table of Rank of costs are given below.

$R(a_{11})=R(a_{12})=R(a_{13})=2, R(a_{14})=1$

$R(a_{21})=10, R(a_{22})=8, R(a_{23})=2, R(a_{24})=4$

$R(a_{31})=7, R(a_{32})=6, R(a_{33})=6, R(a_{34})=8$

Also,Rank of supply:

R(0,2,4,6)=3,R(2,4,9,13)=7,R(2,4,6,8)=5 and

Rank of Demand:

R(1,3,5,7)=4,R(0,2,4,6)=3,R(1,3,5,7)=4,R(1,3,5,7)=4.

Here $\sum a_i=\sum b_j=(4,10,19,27)$ and R(4,10,19,27)=15.

Therefore the problem is balanced fuzzy transportation problem. There exists a fuzzy initial basic feasible solution in crisp form.

Table in Ranking index:

	D1	D2	D3	D4	Supply
S1	2	2	2	1	3
S2	10	8	5	4	7
S3	7	6	6	8	5
Demand	4	3	4	4	15

Using Vogel's Approximation Method, we have

	D1	D2	D3	D4	Supply
S1	2	2	2	1	
	3				3
S2	10	8	5	4	
			3	**4**	7
S3	7	6	6	8	
	1	**3**	**1**		5
Demand	4	3	4	4	15

Since the number of occupied cell having m+n-1=4+3-1=6 and are also independent,there exists a non-degenerate fuzzy basic feasible solution.

$\therefore$ Initial fuzzy transportation minimum cost:

Min Z = Min z^*= 2.3+5.3+4.3+7.1+6.3+6.1=68.

Example 3. *A company has 3 origins O1,O2,O3, and 4 destinations D1,D2,D3,D4. Fuzzy transportation cost for unit product from i^{th} origin to j^{th} destination is c_{ij}:*

	D1	D2	D3	D4	Supply
01	(12.95,13,13,13.05)	(14.95,15,15,15.05)	(15.95,16,16,16.05)	(17.95,18,18,18.05)	(279.95,280,280,280.05)
02	(19.95,20,20,20.05)	(21.95,22,22,22.05)	(10.95,11,11,11.05)	(7.95,8,8,8.05)	(329.95,330,330,330.05)
03	(18.95,19,19,19.05)	(24.95,25,25,25.05)	(16.95,17,17,17.05)	(10.95,11,11,11.05)	(399.95,400,400,400.05)
Demand	(299.95,300,300,300.05)	(249.95,250,250,250.05)	(279.95,280,280,280.05)	(179..95,180,180,180.05)	

Solve Fuzzy Transportation Problem using Robust Ranking

FUZZY RUSSELL'S METHOD

Fuzzy Russell's Method is used for finding initial basic feasible solution for fuzzy transportation problem.The method of Fuzzy Russell's Method are given below:

Procedure for Fuzzy Russell's Method:

Calculate the quantities $\tilde{U}_i$, $\tilde{V}_j$ and $\tilde{C}_{ij} - \tilde{U}_i - \tilde{V}_j$ for all *i, j*.,

where $\tilde{C}_{ij}$ is fuzzy cost of transportation one unit of production i^{th} source to j^{th} destination .

$\tilde{U}_i = \max_{1 \le j \le n} \{ \tilde{C}_{ij} \}$ for i=1,2,3,…m

$\tilde{V}_j = \max_{1 \le i \le m} \{\tilde{C}ij\}$ for j=1,2… n.

and $\Delta_{ij} = \tilde{C}_{ij} - \tilde{U}_i - \tilde{V}_j$ for all *i, j*.

Select the variables x_{ij} having the the most negative value of $\Delta_{ij} = \tilde{C}_{ij} - \tilde{U}_i - \tilde{V}_j$. If there are ties in the values of $\tilde{\Delta}_{ij}$, select x_{ij} having the smallest unit cost $\tilde{C}_{ij}$. If again there are lies in the value of $\tilde{C}_{ij}$, select x_{ij} having the largest

amount of remaining source supply or destination demand. Set the activity level of x_{ij} equal to smaller value between the source supply $\tilde{a}_i$ and destination demand $\tilde{b}_j$.

Next step, subtract x_{ij} from supply $\tilde{a}_i$ or demand $\tilde{b}_j$ found in above . Eliminate the row or column from the transportation table so that results in zero supply or destination demand after subtraction .
Stop if all $\tilde{a}_i$ and $\tilde{b}_j$ are zero , otherwise start from beginning.

Test of Optimality:

For occupied cell,find out a set of numbers $\tilde{U}_i, \tilde{V}_j$ satisfying $\tilde{U}_i + \tilde{V}_j = \tilde{C}_{ij}$ and for each unoccupied cell, find out the following cases

(i)First we check the value of $\Delta_{ij} = \tilde{C}_{ij} - \tilde{U}_i - \tilde{V}_j$. If its positive,the solution is optimal and a unique solution

(ii) If all $\Delta_{ij} > 0$ with atleast one $\Delta_{ij} = 0$, then the solution is optimal but not unique.

(iii) If $\Delta_{ij} = \tilde{C}_{ij} - \tilde{U}_i - \tilde{V}_j$ is negative for atleast one i,j; then the solution is not optimal.

For this case (iii) , to find a new fuzzy bfs, we include in it that the unoccupied cell for which Δ_{ij} is minimum(negative). We allocate maximum amount possible in this cell and make one previous occupied cell empty. By allocating the cell with minimum cell evaluation to decrease the total cost.
For he new set of fuzzy bfs obtained, repeat the procedure until a fuzzy optimal solution is obtained.

Numerical Example:

in which Δ_{ij} is minimum(negative). We allocate maximum amount possible in these cells empty. In such a way,

Consider the fuzzy transportation problem.

	D_1	D_2	D_3	D_4	a_i
O_1	(1,2,3,4)	(1,3,4,6)	(9,11,12,14)	(5,7.8,11)	(1,6,7,12)
O_2	(0,1,2,4),	(-1,0,1,2)	(5,6,7,8)	(0,1,2,3)	(0,1,2,3)
O_3	(3,5,6,8)	(5,8,9,12)	(12,15,16,19)	(7,9,10,12)	(5,10,12,15)
b_j	(5,7,8,10)	(-1,5,6,10)	(1,3,4,6)	(1,2,3,4)	(6,17,21,30)

Solution:

Here, the problem is balanced fuzzy transportation problem. There exists a fuzzy initial basic feasible solution.
Now applying the fuzzy Russell's method for this problem.
U_1=(9,11,12,14) , U_2=(5,6,7,8), U_3=(12,15,16,19)

V_1=(3,5,6,8), V_2=(5,8,9,12), V_3=(12,15,16,19), V_4=(7,9,10,120)
Δ_{12}= (-25,-18,-15,-8), Δ_{11} =(-21,-16,-13,-8) , Δ_{33} =(_12,-15,-16,-19), Δ_{13} =(-24,-17,-14,-7)

HereΔ_{12} is most negative. x_{12}=(-1,5,6,10), c_{12}=(1,3,4,6)
Eliminating col.2, New
U_1=(9,11,12,14), U_2=(5,6,7,8) , U_3=(12,15,16,19), Δ_{33}=(-33,-23,-20,-12) is most negative.
C_{33}=(12,15,16,19),x_{33}=(1,3,4,6)
Next : c_{34}=(7,9,10,12),x_{34}=(1,2,3,4)
And c_{11}=((1,2,3,4), x_{11}=(-9,0,2,13); c_{21}=(0,1,2,4),,x_{21}=(0,1,2,3) and
c_{31}=(3,5,6,8),x_{31}=(-1,6,9,14)

	D_1	D_2	D_3	D_4	a_i
O_1	(1,2,3,4) **(-9,0,2,13)**	(1,3,4,6) **(-1,,5,6,10)**	(9,11,12,14)	(5,7.8,11)	(1,6,7,12)
O_2	(0,1,2,4), **(0,1,2,3)**	(-1,0,1,2)	(5,6,7,8)	(0,1,2,3)	(0,1,2,3)
O_3	(3,5,6,8) **(-1,6,9,14)**	(5,8,9,12)	(12,15,16,19) **(1,3,4,6)**	(7,9,10,12) **(1,2,3,4)**	(5,10,12,15)
b_j	(5,7,8,10)	(-1,5,6,10)	(1,3,4,6)	(1,2,3,4)	(6,17,21,30)

Minimum Z= $c_{33}.x_{33}+c_{12}.x_{12}+ c_{34}.x_{34}+c_{11}.x_{11}+c_{21}.x_{21}+c_{31}.x_{31}$

=(12,15,16,19).(1,3,4,6)+(1,3,4,6).(-1,5,6,10) +((7,9,1012).(1,2,3,4)

+(1,2,3,4).(-9,0,2,13)+(0,1,2,4).(0,1,2,3)+(3,5,6,8).(-1,6,9,14)

=(19,45,60,114)+(-6,15,24,60)+(7,20,30,48)+(-36,0,6,52)+(0,1,4,12)+(-8,30,54,112)

=(-24,111,178,398)

Using α-cut and Ranking method,
The crisp value of Min Z
=160.75
In this problem, for the test of optimality, we compute all $\Delta_{ij}>0$, are positive, the solution under test is optimal.

CONCLUSION

In this paper the fuzzy transportation problem with fuzzy trapezoidal numbers has been transformed into crips transportation problem using Robust's ranking method. By example we can have optimal solution as well as crisp and fuzzy optimal cost. We also proposed Fuzzy Russell's method to find fuzzy feasible solution using Yoger's ranking method. Here, the computation of Russell's method is easier and take little time compare to Robust method. These methods can also be tried to solve other types of problems ,like project schedules,assignment problem and network flow problem.

ACKNOWLEDGMENTS

The author would like to heartily thank to Prof. Manoranjan Maity, Ex-Professor, Vidyasagar Univ., West Bengal. India for his careful reading and helpful suggestions to complete this paper.

REFERENCES

1. B. Hadi, "An approach for solving fuzzy transportation problem," *Application Mathematical Sciety*, **5**, 1549-1566 (2011).
2. S. Chanas and D.Kuchta, "A concept of solution of the transportation problem with fuzzy cost coefficients," *Fuzzy Sets and Systems* **82**,299-305(1992).
3. A.N. Gani and R.K.Abdul, "Two stage fuzzy transportation problem," *Journal. of Physical Sciences*, **10**, 63-69 (2006),.
4. S. Ismail Mohideen and A.Kumar, "A comparative Study on Transportation Problem in Fuzzy Environment," *International Journal of Mathematical Research*, ISSN 0976-5840, 151-158 (2010).
5. P. Kumari, "A comparative study of optimization methods for fuzzy transportation problems," *International Journal Science Engineering Research*, **5,** 76-80(2014).
6. S. Narayanmoorthy, S. Saranya and S. Maheswari ,"A method for solving fuzzy transportation problem using fuzzy Russell's method," *Journal Intelligent Systems and Application*.**02** 71-75(2013).
7. P. Pandianand and G. Natarajan, "A new algorithm for finding a fuzzy optimal solution for fuzzy transportation problems", *Application of Mathematical Science*, **4,** 79-90 (2010).
8. Shugani Poonam, S.H. Abbas and V.K. Gupta, "Fuzzy transportation problem of trapezoidal numbers with α-cut and ranking technique," *Inernational Journal of Fuzzy Mathematics. Systems.* **2** 263-267(2012).
9. Shiang-tai Liu and Chiang Kao, "Solving fuzzy transportation problems based on extension principle," *European Journal of Operational Research*, **153,** 661-674(2004).

10. D. Stephen and K. Palanivel, "The transportation problem in fuzzy Enviroment," *International Journal of Algorithms, Computing and Mathematics*, **2**(3) (2009).
11. H. Tanaka, H. Ichihashi and K. Asai,"A formulation of fuzzy linear Prog. Based on comparison of fuzzy numbers," *Control and Cybernetics*, **13** 185-194(1984).
12. R.R. Yagaer,"A characterization of the extension principle," *Fuzzy Sets and Systems* **18**,205-217(1986).
13. L.A. Zedah, Fuzzy sets, *Information and Control* **8** 338-353(1965).

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 55- 58.

The Efforts to Improve the Ability of Prospective Mathematics Teachers in Developing Students' Mathematical Problem Solving Skills through Micro Teaching Course

Elly Arliani

Department of Mathematics Education, Yogyakarta State University
Email: arlianielly@uny.ac.id

Abstract. The most important part in learning mathematics at school is mathematical problem solving. Mathematical problem solving skills are needed in various fields of occupations and in daily life. Therefore, this skills need attention from teachers and prospective mathematics teachers. Before joining internship, which is the actual teaching practices in the classroom, the prospective mathematics teachers must firstly pass the Micro Teaching course. Micro Teaching course aims at training prospective teachers to have the basic teaching skills. For prospective mathematics teachers, beside ensuring that they have basic teaching skills, they are also be prepared by attaining the ability to implement the learning in order to develop students' mathematical problem solving skills. Based on the author's experience in conducting Micro Teaching course since 2008 and related literature, the author presents this article to describe various efforts which can be done by prospective mathematics teachers in developing students' mathematical problem solving skills through Micro Teaching course.

Keywords: mathematical problem solving, micro teaching course.

INTRODUCTION

Micro Teaching in Mathematics Education Study Program, Faculty of Mathematics and Natural Sciences, Yogyakarta State University, is given to students in semester VI who will carry out the teaching practice at schools. In this course, the students are equipped with the skills required for being a teacher of mathematics. As a prospective mathematics teacher, having only math skills and the ability to teach students are not enough. The ability to develop students' mathematical problem solving skills at schools has to be mastered by the prospective mathematics teachers. Problem solving skills are very necessary for a student to be applied in daily life. Someone who has good problem solving skills will build a resilient personality, do not give up easily, and is able to think through the various alternative solutions which are useful for everyday life and working later. Thus, to be a professional math teacher, a person needs to have knowledge about how to teach students, math skills, and the ability to develop students' mathematical problem solving skills.

DISCUSSION

Micro Teaching Course

Micro Teaching Course in Mathematics Education Study Program Yogyakarta State University is implemented after the prospective mathematics teacher take several compulsory mathematics subjects, several courses of education and some related subjects in mathematics learning. Some subjects of mathematics learning includes planning the mathematics learning, making instruments (lesson plans, student worksheets, and assessments related to competencies expected from students after participating in the learning process), compiling learning media, assessing student learning outcomes, evaluating the teaching and learning process, and psychological of learning mathematics. Micro Teaching is conducted in one semester with 8 – 10 prospective mathematics teachers each class under the guidance of two Mathematics Education lecturers. During the course, prospective mathematics teachers take turns in teaching practice as a mathematics teacher. Each prospective

mathematics teacher has the opportunity at least 5 times to practice as a teacher in the class. When a prospective mathematics teacher becomes a teacher, the others act as students involving in the teaching and learning process and observers assessing the process of teaching. Therefore, each prospective mathematics teacher minimal has experience to be a teacher for 5 times and to be a student and also an observer for 35 times. By participating in Micro Teaching course, prospective mathematics teachers are ready to implement their knowledge of teaching and learning in the real classroom.

Micro Teaching course aims to train the prospective mathematics teachers to implement teaching and learning in the classroom. They are taught to plan the learning of mathematics school with a variety of topics and to implement an educational learning. Related to the learning implementation, the prospective mathematics teachers are also trained to get students' attention when the teaching begins, to link learning materials with the knowledge of mathematics that students have, to motivate students to enjoy learning mathematics, to train students to have problem solving skills, to appreciate students who demonstrate a willingness and show the achievement in learning, to guide students who have low ability and achievement patiently, and to close the teaching and learning process.

During the course, both of the lecturers give the prospective mathematics teachers input or comments after they finish the teaching practice. The comments are given not only for the prospective mathematics teacher who takes a role as a mathematics teacher but also for the prospective mathematics teachers who act as the students in the class in order to provide them with a chance to learn and improve their ability and skills in implementing an interactive learning. The prospective mathematics teachers are invited to give comments to their friend who becomes the mathematics teacher during the practice. In addition, they are also asked to do a self-evaluation to train them to handle their weaknesses and to maintain and improve their strengths and abilities.

Problem Solving Skills

Problem solving is the most important part in learning mathematics at school. According to DiMatteo and Lester [1], problem solving is almost similar to doing mathematics. Moreover, Lester and Kehle in [4] characterize problem solving as an activity which involves students in various cognitive behaviors including accessing and using knowledge and experience they have ever got. Successful problem solving involves the coordination of previous knowledge, experience, representation and inference pattern which are normally used and the intuition in attempt to generate new representation and inference pattern [4]. Therefore, the prospective mathematics teachers should have problem solving skills and they should have the opportunity to learn about how to train the students to have good problem solving skills.

The National Council of Teachers of Mathematics (NCTM) in [5] states that problem solving has to be the focus of mathematics learning. In NCTM point of view, by having good problem solving skills in mathematics, students would acquire ways of thinking, habit of persistence and curiosity, and confidence in unfamiliar situation which will serve them well outside the mathematics classroom. Being a good problem solver can lead to great advantages in everyday life and at the work places [5]. According to Polya [6], there are four stages to solve problems i.e. (1) understand the problem, (2) make a plan, (3) carry out the plan, and (4) look back at the completed solutions. NCTM in [5] states a good problem solver tends to naturally and carefully analyze situations in mathematical terms and to overcome problems based on the situation they see. Based on the above, the students who have good problem solving skills are expected to have good fighting spirits, independent, and able to overcome the various problems in the daily life and the world of work.

Improving the Prospective Mathematics Teachers Ability in Developing Students' Mathematical Problem Solving Skills through Micro Teaching Course

Before the prospective mathematics teachers are trained to develop their mathematical problem solving skills in Micro Teaching course, they should know and have the basic of problem solving skills. Problem solving skills were acquired during they took some courses of mathematics and learning of mathematics, that is before they have a lecturing of Micro Teaching course. According to the NCTM [5], someone who has already had good problem solving skills, he/she is able to: 1) build new mathematical knowledge through problem solving, 2) solve problems that arise in mathematics and in other contexts, 3) apply and adapt a variety of appropriate strategies to solve problems, and 4) monitor and reflect on the process of mathematical problem solving. Therefore, the prospective mathematics teachers who have already had mathematical problem solving skills need to be trained to develop their students' mathematical problem solving skills. This ability is enhanced when they practice implementing the teaching and learning with their friends.

There are some important things that the prospective mathematics teachers forget in their teaching practices, particularly in solving the problem. The most common thing they forget is to patiently motivate students. They tend to directly provide the students with assistance and give solution without trying to elaborate the students' ability to find their own solutions. In this regard, Polya in [6] said that one of the most important tasks of the teacher is to help his/her students. This task is not quite easy since it demands time, practice, devotion, and principles. The students should have much experiences of independent work. But if he is left alone with his problem without any help or with insufficient help, he may make no progress at all. If the teacher helps too much, nothing is left for the student. The teacher should help appropriately so that the students will have a reasonable share of the work [6]. Therefore, prospective mathematics teachers need to be trained how to encourage students to always have passion in solving problems, to be independent, and not to give up easily, without fettering their creativity.

Then, what problems should be given to the student? It has also become the concern in Micro Teaching course. According to Treffinger and Isaksen [7], students should have opportunities to work on real-life problems and challenges. The real problems in students' daily will challenge them to solve it. Problems that are not real often make students confused and will not be challenging for them. The prospective mathematics teachers need to have more practices on this matter, because sometimes the problems they show often does not make sense. By taking Micro Teaching course, they are expected to make real and challenging cases or problems to solve.

Furthermore, another thing that is often forgotten by prospective mathematics teachers is how to appreciate the efforts of students who have demonstrated learning progress. Isoda in [3] said that to teach mathematics with these kinds of mathematical activities, we do not only ask children to solve the task given by the teacher, but also give the children opportunities to consider what they would like to do next based on their expectation and appreciate their activities as they begin to learn how to develop mathematics for themselves. Therefore, in developing students' problem solving skills, prospective mathematics teachers should be patient and do not rush to help the students find a solution to the problem they are trying to solve. Through Micro Teaching course, prospective mathematics teachers are trained not to hinder the creativity of students. In addition, the reward in the form of expressions and speech should be considered by prospective mathematics teachers to achieve the success of their students and to motivate the students to always try to find a solution with their capabilities.

Then, to increase students' mathematical problem solving skills, prospective mathematics teachers should allow students to discuss what they know, to plan together for the best possible solutions, implement the agreed plan, and review the results obtained. This is in line with Heller et al. [2] who said that group problem solutions were significantly better than those produced by the best problem solver from each group on matched individual problems, particularly with respect the qualitative analysis of the problems. In cooperative groups, student can share conceptual and procedural knowledge and argument roles, and request clarification, justification, and elaboration from one another, so a better solution emerges than that could be achieved by individuals working alone [2]. Working with groups of students in solving problems will greatly assist prospective mathematics teachers in terms of time, i.e. it will need less time in solving the problems. Then, the smarter students can help their friends who have difficulty and it will help the teacher to do the work. Another benefit of group activities is that there is interaction between students which allows more students achieve learning progress and more alternative solutions that they have acquired.

CONCLUSION

A good problem solver is able to build new knowledge in mathematics through problem solving, to overcome the problems that arise in mathematics and other contexts, to apply and adapt a variety of appropriate strategies to solve problems, and to monitor and reflect on the process of solving mathematical problems. In the activity of problem solving, there are four stages to solve problems i.e. understanding the problem, making a plan, carrying out the plan, and reviewing the completed solutions. The efforts that can be made to improve the ability of prospective mathematics teachers in developing the students' mathematical problem solving skills are to train them to make real-life and challenging problems, to provide opportunities for students to work in groups in order to obtain the best solution, to give opportunities for students to develop their creativity using their experiences and knowledge in improving problem solving strategies, and to appreciate any progress that has been achieved by students.

REFERENCES

1. R. W. DiMatteo and F. K. Lester Jr., "The Role of Problem Solving in the Secondary School Mathematics Classroom" in *Teaching and Learning Mathematics*. Translating Research for Secondary School Teachers (National Council of Teachers of Mathematics, Reston, Va, 2010).
2. P. Heller, R. Keith, and S. Anderson, "Teaching Problem Solving through Cooperative Grouping. Part 1: Group versus Individual Problem Solving. *Journal of American Association of Physics Teachers*. Vol. 60 (7), 627-636 (1992).
3. M. Isoda, "Problem Solving Approach to Develop Mathematical Thinking" in *Mathematical Thinking: How to Develop It in the Classroom*. Monographs on Lesson Study for Teaching Mathematics and Sciences, Vol.1, edited by M. Isoda and S. Katagiri (World Scientific, Singapore, 2012), pp.1-28.
4. F. K. Lester, Jr, and P. E. Kehle, "From Problem Solving to Modeling: The Evolution of Thinking about Research on Complex Mathematical Activity," in *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching* , edited by R. Lesh and H. M. Doerr (Lawrence Erlbaum Associates, Mahwah, 2003), pp.501–517.
5. National Council of Teachers of Mathematics, *Principles and Standard for School Mathematics* (NCTM, Reston, Va., 2000).
6. G. Polya, *How to Solve It: A New Aspect of Mathematical Method*, Second Edition (Doubleday Anchor Books, New York, 1973).
7. D. J. Treffinger and S.G. Isaksen, Creative problem solving: the history, development, and implication for gift education and talent development (*Journal of Gift Child Quarterly*. Vol.49. No.4, 342-353, 2005).

Proceedings of "The 7^{th} SEAMS-UGM Conference 2015"
Topic, pp. 59–66.

A Note on Diagonalization of Matrices over A Symmetrized Max Plus Algebra

Gregoria Ariyanti*, Ari Suparwanto† and Budi Surodjo**

*Department of Mathematics Education, University of Widya Mandala Madiun, Indonesia
ariyanti_gregoria@yahoo.com
†Department of Mathematics, University of Gadjah Mada, Yogyakarta, Indonesia
ari_suparwanto@ugm.ac.id
**Department of Mathematics, University of Gadjah Mada, Yogyakarta, Indonesia
surodjo_b@ugm.ac.id

Abstract. The system max plus algebra does not have an additive inverse. Therefore,some equations do not have a solution. For example, the equation $3 \oplus x = 2$ has no solution since there is no x such that $\max(3,x) = 2$. One way of trying to solve this problem is to extend the max plus algebra to a larger system which will include additive inverse in the same way that the natural numbers were extended to the larger system of integers. The extended system that is larger than max plus means the symmetrized max plus algebra. The symmetrized max plus algebra is an algebraic structure which is a commutative semiring, has a zero element $\varepsilon = -\infty$, the identity element e = 0, and an additively idempotent.
Motivated by the previous study as in conventional linear algebra, a matrix over a symmetrized max plus algebra that are not a diagonal matrix can be changed into a diagonal matrix by diagonalization process. This paper will describe the condition for a matrix over a symmetrized max plus algebra can be diagonalizable.

Keywords: symmetrized max plus algebra, diagonal matrix, diagonalization, diagonalizable

INTRODUCTION

The system max plus algebra lacks an additive inverse. Therefore, some equations do not have a solution. For example, the equation $3 \oplus x = 2$ has no solution since there is no x such that $\max(3,x) = 2$. De Schutter and De Moor in [5] and Singh in [7] state that one way of trying to solve this problem is to extend the max plus algebra to a larger system which will include and additive inverse in the same way that the natural numbers were extended to the larger system of integers. Therefore, we have that the system $(\mathbf{S}, \oplus, \otimes)$ is called the symmetrized max plus algebra and $\mathbf{S} = \mathbf{R}_{\varepsilon}^2/\mathscr{B}$ with $\mathscr{B}$ is an equivalence relation.

In conventional algebra, diagonal matrix is one form of a matrix with all entries zero except the main diagonal. The determinant of a diagonal matrix is the product of the entries on the main diagonal. For any diagonal matrix, the main diagonal entries of the matrix are the eigenvalues of matrix [1].

Some matrix which is not a diagonal matrix can be transformed into a diagonal matrix by diagonalization process. Diagonalization of a matrix is a process of the matrix A which finds a diagonal matrix D which is similar to the matrix A. A $n \times n$ matrix over a field can be diagonalized, if there are n eigen vectors of matrix A which are linear independent. A matrix over a commutative ring can be diagonalized if the union of all eigenspaces of the matrix contains a number of linearly independent vectors that can construct a commutative ring R.

De Schutter in [6] prove the existence of max-algebraic analogues of two basic matrix decompositions from linear algebra: the singular value decomposition and the QR decomposition. Motivated by the singular value decomposition over a max plus algebra, in this paper we will describe diagonalization over a symmetrized max plus algebra. We will use diagonalization to a explored the eigen value decomposition over a symmetrized max plus algebra. In [3] and [6], a matrix over a symmetrized max plus algebra can be characterized by its determinant, we will explore a diagonalization matrix over a symmetrized max plus algebra.

SYMMETRIZED MAX PLUS ALGEBRAS

Let the set of all real numbers $\mathbf{R}_\varepsilon = \mathbf{R} \cup \{\varepsilon\}$ with $\varepsilon := -\infty$ and $e := 0$. For all $a, b \in \mathbf{R}_\varepsilon$, the operations $\oplus$ and $\otimes$ are defined as follows:

$$a \oplus b = max(a,b) \text{ and } a \otimes b = a + b$$

and then, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is called a max plus algebra.

Definition 1 *[5] Let $u = (x,y), v = (w,z) \in \mathbf{R}_\varepsilon^2$.*
1. Two unary operators $\ominus$ and $(.)^\bullet$ are defined as follow :

$$\ominus u = (y,x) \text{ and } u^\bullet = u \oplus (\ominus u)$$

2. An element u is called balances with v, denoted by $u \nabla v$, if

$$x \oplus z = y \oplus w.$$

3. A relation $\mathscr{B}$ is defined as follows :

$$(x,y)\mathscr{B}(w,z) \text{ if } \begin{cases} (x,y)\nabla(w,z), & if \ \ x \neq y \ \ and \ \ w \neq z \\ (x,y) = (w,z), & otherwise \end{cases}$$

Because $\mathscr{B}$ is an equivalence relation, we have the set of factor $\mathbf{S} = \mathbf{R}_\varepsilon^2 / \mathscr{B}$ and the system $(\mathbf{S}, \oplus, \otimes)$ is called the symmetrized max plus algebra, with the operations of addition and multiplication on $\mathbf{S}$ is defined as follows

$$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a \oplus c, b \oplus d)}$$

$$\overline{(a,b)} \otimes \overline{(c,d)} = \overline{(a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)}$$

for $\overline{(a,b)}, \overline{(c,d)} \in \mathbf{S}$.
The system $(\mathbf{S}, \oplus, \otimes)$ is a semiring, because $(\mathbf{S}, \oplus)$ is associative, $(\mathbf{S}, \otimes)$ is associative, and $(\mathbf{S}, \oplus, \otimes)$ satisfies both the left and right distributive.

Lemma 1 *[5] Let $(\mathbf{S}, \oplus, \otimes)$ be the symmetrized max plus algebra. Then the following statements holds.*

1. *$(\mathbf{S}, \oplus, \otimes)$ is commutative*
2. *An element $\overline{(\varepsilon, \varepsilon)}$ is a zero element and an absorbent element.*
3. *An element $\overline{(e, \varepsilon)}$ is an identity element.*
4. *$(\mathbf{S}, \oplus, \otimes)$ is an additively idempotent.*

The system $\mathbf{S}$ is divided into three classes :

- $\mathbf{S}^\oplus$ consists of all positive elements or
  $\mathbf{S}^\oplus = \{\overline{(t,\varepsilon)} | t \in \mathbf{R}_\varepsilon\}$ with $\overline{(t,\varepsilon)} = \{(t,x) \in \mathbf{R}_\varepsilon^2 | x < t\}$
- $\mathbf{S}^\ominus$ consists of all negative elements or
  $\mathbf{S}^\ominus = \{\overline{(\varepsilon,t)} | t \in \mathbf{R}_\varepsilon\}$ with $\overline{(\varepsilon,t)} = \{(x,t) \in \mathbf{R}_\varepsilon^2 | x < t\}$
- $\mathbf{S}^\bullet$ consists of all balanced elements or
  $\mathbf{S}^\bullet = \{\overline{(t,t)} | t \in \mathbf{R}_\varepsilon\}$ with $\overline{(t,t)} = \{(t,t) \in \mathbf{R}_\varepsilon^2\}$

Because $\mathbf{S}^\oplus$ isomorphic to $\mathbf{R}_\varepsilon$, so it will be shown that for $a \in \mathbf{R}_\varepsilon$, can be expressed by $\overline{(a,\varepsilon)} \in \mathbf{S}^\oplus$.
Furthermore, it is easy to verify that for $a \in \mathbf{R}_\varepsilon$ we have :

$$a = \overline{(a,\varepsilon)} \text{ with } \overline{(a,\varepsilon)} \in \mathbf{S}^\oplus$$

$$\ominus a = \ominus\overline{(a,\varepsilon)} = \overline{\ominus(a,\varepsilon)} = \overline{(\varepsilon,a)} \ \ with \ \ \overline{(\varepsilon,a)} \in \mathbf{S}^\ominus$$

$$a^\bullet = a \ominus a = \overline{(a,\varepsilon)} \ominus \overline{(a,\varepsilon)} = \overline{(a,\varepsilon)} \oplus \overline{(\varepsilon,a)} = \overline{(a,a)} \in \mathbf{S}^\bullet$$

Lemma 2 *For a,b* $\in \mathbf{R}_\varepsilon$,$a \ominus b = \overline{(a,b)}$.

PROOF.

$$a \ominus b = \overline{(a,\varepsilon)} \ominus \overline{(b,\varepsilon)} = \overline{(a,\varepsilon)} \oplus \overline{(\varepsilon,b)} = \overline{(a,b)}$$

Lemma 3 *For* $\overline{(a,b)} \in \mathbf{S}$ *with a,b* $\in \mathbf{R}_\varepsilon$, *the following statements hold :*

1. *If a>b then* $\overline{(a,b)} = \overline{(a,\varepsilon)}$
2. *If a<b then* $\overline{(a,b)} = \overline{(\varepsilon,b)}$
3. *If a=b then* $\overline{(a,b)} = \overline{(a,a)}$ *or* $\overline{(a,b)} = \overline{(b,b)}$

PROOF.

1. For $a > b$ we have that $a \oplus b = a$. In other words, $a \oplus \varepsilon = a \oplus b$. The result that $(a,b)\nabla(a,\varepsilon)$. So it follows that $(a,b)\mathscr{B}(a,\varepsilon)$. Therefore $\overline{(a,b)} = \overline{(a,\varepsilon)}$.
2. For $a < b$ we have that $a \oplus b = b$. In other words, $a \oplus b = b \oplus \varepsilon$. The result that $(a,b)\nabla(\varepsilon,b)$. So it follows that $(a,b)\mathscr{B}(\varepsilon,b)$. Therefore $\overline{(a,b)} = \overline{(\varepsilon,b)}$.

Corollary 1 *For* $a,b \in \mathbf{R}_\varepsilon$,

$$a \ominus b = \begin{cases} a, \ if \ a > b \\ \ominus b, \ if \ a < b \\ a^\bullet, \ if \ a = b \end{cases}$$

MATRICES OVER A SYMMETRIZED MAX PLUS ALGEBRA

Let **S** the symmetrized max plus algebra, *n* a positive integer greater than 1 and $M_n(\mathbf{S})$ is the set of all *nxn* matrices over **S**. Operations $\oplus$ and $\otimes$ for matrices over a symmetrized max plus algebra is defined :

$$C = A \oplus B \Rightarrow c_{ij} = a_{ij} \oplus b_{ij}$$

$$C = A \otimes B \Rightarrow c_{ij} = \bigoplus_l a_{il} \otimes b_{lj}$$

Zero matrix *nxn* over **S** is ε_n with $(\varepsilon_n)_{ij} = \varepsilon$ and identity matrix *nxn* over **S** is E_n with

$$[E_n]_{ij} = \begin{cases} e, \ if \ i = j \\ \varepsilon, \ if \ i \neq j \end{cases}$$

Definition 2 *We say that the matrix A*$\in M_n(\mathbf{S})$ *is invertible over* **S** *if*

$$A \otimes B \nabla E_n \ and \ B \otimes A \nabla E_n$$

for any $B \in M_n(\mathbf{S})$.

Definition 3 *[3] Let a matrix* $A \in M_n(\mathbf{S})$. *The determinant of A is defined by*

$$detA = \bigoplus_{\sigma \in \mathscr{S}_n} sgn(\sigma) \otimes (\bigotimes_{i=1}^n A_{i\sigma(i)})$$

with $\mathscr{S}_n$ *is the set of all permutations of* $\{1,2,...,n\}$, *and*

$$sgn(\sigma) = \begin{cases} 0, \ if \ \sigma \ is \ even \ permutation \\ \ominus 0, \ if \ \sigma \ is \ odd \ permutation \end{cases}$$

Note that the operator "∇" and the systems of max-linear balances hold :

Lemma 4 *[5]*

1. $\forall a,b,c \in \mathbf{S}, a \ominus c \nabla b \Leftrightarrow a \nabla b \oplus c$

2. $\forall a,b \in \mathbf{S}^{\oplus} \cup \mathbf{S}^{\ominus}, a\nabla b \Rightarrow a = b$
3. *Let $A \in M_n(\mathbf{S})$. The homogeneous linear balance $A \otimes x\nabla\varepsilon_{nx1}$ has a non trivial solution in $\mathbf{S}^{\oplus}$ or $\mathbf{S}^{\ominus}$ if and only if $det(A)\nabla\varepsilon$.*

Proposition 1 *[5] Let $A \in M_n(\mathbf{S})$. If $det(\lambda \otimes E_n \ominus A)\nabla\varepsilon$ we have*

$$\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}} \nabla\varepsilon$$

with

$$a_k = (\ominus 0)^{\otimes^k} \otimes \bigoplus_{\varphi \in C_n^k} detA_{\varphi\varphi}$$

or

$$a_k = (\ominus 0)^{\otimes^k} \bigoplus_{i_1,\ldots,i_k \in C_n^k} \bigoplus_{\varphi \in P_k} sgn(\sigma) \otimes \bigotimes_{r=1}^{k} a_{i_r i_{\sigma(r)}}$$

for k = 1,2,...,n.

DIAGONALIZATION MATRICES OVER A SYMMETRIZED MAX PLUS ALGEBRA

The symmetrized max plus algebra is an idempotent semiring, so in order to define rank, linear combination, linear dependence, and independence we need definition of a semimodule. A semimodule is essentially a linear space over a semiring.

A non diagonal matrix over a symmetrized max plus algebra can be changed a diagonal matrix. This process is called diagonalization.

Definition 4 *Let $a_1, a_2, \ldots, a_m \in M_{nx1}(\mathbf{S})$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbf{S}$.*
The expression $\bigoplus_{i=1}^{m} \alpha_i \otimes a_i$ is called a linear combination of $\{a_1, a_2, \ldots, a_m\}$.

Definition 5 *A set of vectors $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,\ldots,m\}$ is said to be a linearly independent set whenever the only solution for the scalars α_i in $\bigoplus_{i=1}^{m} \alpha_i \otimes a_i \nabla\varepsilon_{nx1}$ is the trivial solution $\alpha_i = \varepsilon$.*

The relation between linear dependence and linear combination are given in the following theorem.

Theorem 1 *If the set of vectors $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,\ldots,n\}$ is linearly dependent then one of the vectors can be presented as a linear combination of the other vectors in the set.*

PROOF. Let $\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla\varepsilon_{n\times 1}$.
Because $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,\ldots,n\}$ is a linearly dependent set, without loss of generality, we can take $\alpha_1 \neq \varepsilon_{n\times 1}$.
So, there is a scalar $\alpha_1^{\otimes^{-1}}$ such that

$$\alpha_1^{\otimes^{-1}} \otimes (\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n) \nabla \alpha_1^{\otimes^{-1}} \otimes \varepsilon_{n\times 1}$$

$$a_1 \oplus \alpha_1^{\otimes^{-1}} \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_1^{\otimes^{-1}} \otimes \alpha_n \otimes a_n \nabla \varepsilon_{n\times 1}$$

$$a_1 \nabla \beta_2 \otimes a_2 \oplus \beta_3 \otimes a_3 \oplus \ldots \oplus \beta_n \otimes a_n \;\; or \;\; a_1 \nabla \bigoplus_{i=2}^{n} \beta_i \otimes a_i$$

with $\beta_i = \ominus \alpha_1^{\otimes^{-1}} \otimes \alpha_i, i = 2,3,\ldots,n$.
This leads to a characterization of linear dependent in term of determinants.

Theorem 2 *Let $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,\ldots,n\}$ be a vector set.*
Construct a matrix A such that $A = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$.
The set of vectors $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,\ldots,n\}$ are linearly dependent if and only if $det(A)\nabla\varepsilon$.

PROOF. ($\Rightarrow$) Because $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,...,n\}$ is a linearly dependent set, this implies that one of the vectors (without loss of generality, let's say its the vector a_1) can be presented as a linear combination of the other vectors in the set. Or,

$$a_1 \nabla \beta_2 \otimes a_2 \oplus \beta_3 \otimes a_3 \oplus ... \oplus \beta_n \otimes a_n \;\; or \;\; a_1 \nabla \bigoplus_{i=2}^{n} \beta_i \otimes a_i$$

Next, take matrix A and subtract a_1 with $\bigoplus_{i=2}^{n} \beta_i \otimes a_i$. This results in another matrix (say A') whose last column is a ε vector.

$$A = \begin{pmatrix} a^{\bullet} & a_2 & ... & a_n \end{pmatrix} \nabla \begin{pmatrix} \varepsilon & a_2 & ... & a_n \end{pmatrix}$$

Because $det \begin{pmatrix} \varepsilon & a_2 & ... & a_n \end{pmatrix} \nabla \varepsilon$ so, we now have that $det(A') \nabla \varepsilon$.
It follows from the fact that one of the columns of the matrix being ε, that $det(A) \nabla \varepsilon$.
($\Leftarrow$) Let $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,...,n\}$ is a linearly independent.
We can show that det(A) not balanced with ε.
Construct

$$\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla \varepsilon$$

We have that

$$a_1 \otimes \alpha_1 \oplus a_2 \otimes \alpha_2 \oplus \ldots \oplus a_n \otimes \alpha_n \nabla \varepsilon$$

Consequently, $\begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \nabla \varepsilon$.
Because $\{a_i \in M_{nx1}(\mathbf{S}) | i = 1,2,...,n\}$ is a linearly independent, so we have

$$\alpha_1 = \alpha_2 = ... = \alpha_n = \varepsilon$$

We can see, that homogenous linear balance $A \otimes x \nabla \varepsilon$ has a trivial solution $x = \varepsilon$.
Since it follows from lemma 4 that the homogeneous linear balance $A \otimes x \nabla \varepsilon$ has a non trivial signed solution if and only if $detA \nabla \varepsilon$, so we have det(A) not balanced with ε.
A matrix can be diagonalizable if any invertible matrix P such that $P^{\otimes^{-1}} \otimes A \otimes P \nabla D$ with D matrix diagonal and

$$D = \begin{pmatrix} \lambda_1 & ... & \varepsilon \\ \vdots & \ddots & \vdots \\ \varepsilon & ... & \lambda_r \end{pmatrix}.$$

We have that, if $A \in M_n(\mathbf{S})$, with $\lambda_1, \lambda_2, ..., \lambda_r \in \mathbf{S}$, has different eigen values and $v_1, v_2, ..., v_r \in M_{nx1}(\mathbf{S})$ are eigenvector that corresponding with eigenvalue, so

$$A \nabla P \otimes D \otimes P^{\otimes^{-1}}$$

with D diagonal matrix which

$$D = \begin{pmatrix} \lambda_1 & ... & \varepsilon \\ \vdots & \ddots & \vdots \\ \varepsilon & ... & \lambda_r \end{pmatrix}.$$

Relation between diagonalizable matrix and set of linearly independent vectors are given in the following theorem.

Theorem 3 *A square matrix A is said to be diagonalizable if and only if the matrix A has n linearly independent eigenvectors.*

PROOF. ($\Rightarrow$) Let $A \in M_n(\mathbf{S})$ can be diagonalized.
So that there exists a nonsingular matrix $P = \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix}$ such that $P^{\otimes^{-1}} \otimes A \otimes P \nabla D$ with

$$D = \begin{pmatrix} \lambda_1 & \varepsilon & \dots & \varepsilon \\ \varepsilon & \lambda_2 & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \lambda_r \end{pmatrix}.$$

By theorem 2, we have $\{v_1, v_2, ..., v_r\}$ is linearly independent.
($\Leftarrow$) Let $\{v_1, v_2, ..., v_r\}$ is linearly independent.
Assume $A \otimes v_i \nabla \lambda_i \otimes v_i$ for $i = 1, 2, ..., r$. We have $P = \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix}$
$A \otimes P = \begin{bmatrix} A \otimes v_1 & A \otimes v_2 & \dots & A \otimes v_r \end{bmatrix}$
$\nabla \begin{bmatrix} \lambda_1 \otimes v_1 & \lambda_2 \otimes v_2 & \dots & \lambda_r \otimes v_r \end{bmatrix}$
$= \begin{bmatrix} v_1 \otimes \lambda_1 & v_2 \otimes \lambda_2 & \dots & v_r \otimes \lambda_r \end{bmatrix}$

$$= \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix} \otimes \begin{pmatrix} \lambda_1 & \varepsilon & \dots & \varepsilon \\ \varepsilon & \lambda_2 & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \lambda_r \end{pmatrix} = P \otimes D.$$

Therefore, we have $A \otimes P \nabla P \otimes D$.

Next, we give representation of characteristic equation.

Definition 6 *[5] Let $A \in M_n(\mathbf{S})$.*
For $det(\lambda \otimes E_n \ominus A) \nabla \varepsilon$, we obtain

$$\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}} \nabla \varepsilon \tag{1}$$

with

$$a_k = (\ominus 0)^{\otimes^k} \otimes \bigoplus_{\varphi \in C_n^k} detA_{\varphi\varphi} \tag{2}$$

or

$$a_k = (\ominus 0)^{\otimes^k} \bigoplus_{i_1, \dots, i_k \in C_n^k} \bigoplus_{\varphi \in P_k} sgn(\sigma) \otimes \bigotimes_{r=1}^{k} a_{i_r i_{\sigma(r)}} \tag{3}$$

for k = 1,2,...,n.

Let $A = \begin{pmatrix} \ominus 2 & 1 & \varepsilon \\ 1 & \ominus 0 & 1 \\ \varepsilon & 0 & 2 \end{pmatrix}$. For matrix A, with Proposition 1:
$a_k = (\ominus 0)^{\otimes^k} \otimes \bigoplus_{\varphi \in C_n^k} detA_{\varphi\varphi}$
we have,
$a_1 = (\ominus 0)^{\otimes^1} \otimes \bigoplus_{\varphi \in C_3^1} detA_{\varphi\varphi}$
$a_1 = (\ominus 0) \otimes (detA_{11} \oplus detA_{22} \oplus detA_{33})$
$a_1 = (\ominus 0) \otimes (a_{11} \oplus a_{22} \oplus a_{33}) = \ominus(\ominus 2 \oplus \ominus 0 \oplus 2) = 2^{\bullet}$
With the same process, we have
$a_2 = (\ominus 0)^{\otimes^2} \otimes \bigoplus_{\varphi \in C_3^2} detA_{\varphi\varphi}$
$a_2 = 0 \otimes (detA_{\{1,2\}\{1,2\}} \oplus detA_{\{1,3\}\{1,3\}} \oplus detA_{\{2,3\}\{2,3\}})$
$a_2 = \begin{pmatrix} \ominus 2 & 1 \\ 1 & \ominus 0 \end{pmatrix} \oplus \begin{pmatrix} \ominus 2 & \varepsilon \\ \varepsilon & 2 \end{pmatrix} \oplus \begin{pmatrix} \ominus 0 & 1 \\ 0 & 2 \end{pmatrix}$
$a_2 = ((\ominus 2 \otimes \ominus 0) \ominus (1 \otimes 1)) \oplus ((\ominus 2 \otimes 2) \ominus (\varepsilon \otimes \varepsilon)) \oplus ((\ominus 0 \otimes 2) \ominus (1 \otimes 0)) = (2 \ominus 2) \oplus (\ominus 4 \ominus \varepsilon) \oplus (\ominus 2 \ominus 1) = \ominus 4$
With the same process, we have
$a_3 = (\ominus 0)^{\otimes^3} \otimes \bigoplus_{\varphi \in C_3^3} detA_{\varphi\varphi}$

$$a_3 = \ominus \begin{pmatrix} \ominus 2 & 1 & \varepsilon \\ 1 & \ominus 0 & 1 \\ \varepsilon & 0 & 2 \end{pmatrix} = \ominus\{(4 \oplus \varepsilon \oplus \varepsilon) \ominus (\varepsilon \oplus \ominus 3 \oplus 4)\} = \ominus(4 \ominus 4) = 4^{\bullet}$$

Finally, we have its characteristic equation, that is

$$\lambda^{\otimes^3} \oplus a_1 \otimes \lambda^{\otimes^2} \oplus a_2 \otimes \lambda^{\otimes^1} \oplus a_3 \nabla \varepsilon$$

or

$$\lambda^{\otimes^3} \oplus 2^{\bullet} \otimes \lambda^{\otimes^2} \oplus \ominus 4 \otimes \lambda \oplus 4^{\bullet} \nabla \varepsilon$$

With algorithm in ELCP (*Extended Linear Complementarity Problem*), we have the set of all eigen values $\bar{u} = \kappa \begin{pmatrix} -8 \\ 0 \\ 0 \end{pmatrix}$.

In ELCP, $\bar{u} = \begin{pmatrix} \bar{x} \\ \alpha \end{pmatrix} = \begin{pmatrix} \lambda \\ p \\ \alpha \end{pmatrix}$

Hence, $\lambda = \kappa(-8)$. Let $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = -4$. The associated eigenvectors for A are :

- Let $\lambda = 0$. We have the set of all eigenvectors associated with $\lambda = 0$, that is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \nabla \left\{ \begin{pmatrix} t \\ 1 \otimes t \\ 1^{\bullet} \otimes t \end{pmatrix} | t \in \mathbf{S} \right\}$$

- Let $\lambda = -2$. We have the set of all eigenvectors associated with $\lambda = -2$, that is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \nabla \left\{ \begin{pmatrix} s \\ 1 \otimes s \\ 0^{\bullet} \otimes s \end{pmatrix} | s \in \mathbf{S} \right\}$$

- Let $\lambda = -4$. We have the set of all eigenvectors associated with $\lambda = -4$, that is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \nabla \left\{ \begin{pmatrix} r \\ 1 \otimes r \\ 0^{\bullet} \otimes r \end{pmatrix} | t \in \mathbf{S} \right\}$$

We construct matrix $P = \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix} = \begin{pmatrix} \ominus 1 & \ominus 1 & \ominus 1 \\ \ominus 2 & \ominus 2 & \ominus 2 \\ 2^{\bullet} & (1)^{\bullet} & (1)^{\bullet} \end{pmatrix}$

with $D = \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & -4 & \varepsilon \\ \varepsilon & \varepsilon & -2 \end{pmatrix}$

We have $A \otimes P = \begin{pmatrix} \ominus 2 & 1 & \varepsilon \\ 1 & \ominus 0 & 1 \\ \varepsilon & 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} \ominus 1 & \ominus 1 & \ominus 1 \\ \ominus 2 & \ominus 2 & \ominus 2 \\ 2^{\bullet} & (1)^{\bullet} & (1)^{\bullet} \end{pmatrix} = \begin{pmatrix} 3^{\bullet} & 3^{\bullet} & 3^{\bullet} \\ 3^{\bullet} & 2^{\bullet} & 2^{\bullet} \\ 4^{\bullet} & 3^{\bullet} & 3^{\bullet} \end{pmatrix}$

Hence

$$P \otimes D = \begin{pmatrix} \ominus 1 & \ominus 1 & \ominus 1 \\ \ominus 2 & \ominus 2 & \ominus 2 \\ 2^{\bullet} & (1)^{\bullet} & (1)^{\bullet} \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & -4 & \varepsilon \\ \varepsilon & \varepsilon & -2 \end{pmatrix} = \begin{pmatrix} \ominus 1 & \ominus -3 & \ominus(-1) \\ \ominus 2 & \ominus(-2) & \ominus 0 \\ 2^{\bullet} & (-3)^{\bullet} & (-1)^{\bullet} \end{pmatrix}$$

Because $A \otimes P \nabla P \otimes D$, so $P^{-1} \otimes A \otimes P \nabla D$.

REFERENCES

1. Anton, H.: *Aljabar Linear Elementer* [Silaban, Susila,trans]. 5th ed. Hutauruk R, editor. Jakarta: Erlangga (1987).
2. Baccelli, F., Cohen, G., Olsder, G.J., and Quadrat, J.P. , 2001. *Synchronization and Linearity*. John Wiley and Sons New York, Inc., USA.

3. Gaubert, S. et. al. : Minimal (Max,+) Realization of Convex Sequences. *SIAM J. Control Optim*. Vol. 36. No. 1. pp. 137-147 (1988).
4. Mora, W. et. al. : Invertible Matrices Over Idempotent Semirings. *Chamcuri Journal of Mathematics* volume 1, Number 2, pp. 55-61 (2009).
5. De Schutter, B. and De Moor, B., 1997a. A Note on The Characteristic Equation in The Max-Plus Algebra. *Linear Algebra and Its Applications*, vol. 261, no. 13, pp. 237250.
6. De Schutter, B. and De Moor, B., 1997b. The Extended Linear Complementarity Problem and Its Application in The Max-Plus Algebra. *in Complementarity and Variational Problems: State of the Art* (M.C. Ferris and J.S. Pang, eds.), Philadelphia, Pennsylvania: SIAM, ISBN 0−89871−391−9, pp. 2239
7. Singh, D., Ibrahim, M., and Singh, J.N., 2008. A Note on Symmetrized Max-Plus Algebra. *Journal of Mathematical Sciences and Mathematics Education*. Vol. 5. No.1.

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 67-74.

A Study of Logarithmic Transformation Model in Spatial Empirical Best Linear Unbiased Prediction (SEBLUP) Method of Small Area Estimation

Hazan Azhari Zainuddin[1)], Khairil Anwar Notodiputro[2)] and Kusman Sadik[3)]

[1]*Master Student of Bogor Agricultural University, Email: hazanazhari78@gmail.com*
[2]*Lecturer of Bogor Agricultural University, Email: khairilnotodiputro@gmail.com*
[3]*Lecturer of Bogor Agricultural University, Email: kusmansadik@gmail.com*

Abstract. There have been many studies developed to improve the quality of estimates in small area estimation (SAE). The standard method known as EBLUP (Empirical Unbiased Best Linear Predictor) has been developed by incorporating spatial effects into the model. This modification of the method was known SEBLUP (Spatial EBLUP) since it incorporates the spatial correlations which exist among the small areas. The data obtained (variables of concern) usually have a large variance and tend to have a a nonsymmetric distribution and therefore tend to have nonlinear relationship pattern between concomitant variables and variables of concern. the results showed that the method SEBLUP using logarithmic transformation produces estimator more than the other methods.

Keywords: EBLUP, SAE, SEBLUP

INTRODUCTION

Various surveys are generally designed to estimate population parameters of national level. Problems will arise if the survey would like to obtain information for smaller areas, for example at the provincial level, district level, or sub-district level. The size of the sample at the level of the area is usually so small that the statistics obtained will have a large variance. To overcome this problem, developed a parameter estimation method called small area estimation methods (SAE).

Small area estimation has now become the world's attention statisticians very seriously. There have been many studies were developed both for the improvement of techniques and the development of methods and applications in a variety of cases and real problems faced. Fay and Herriot in 1979 (referred to [6]) was the first researcher to develop a small area estimation based models. Models are developed and then become a reference in the development of small area estimation study further until today.

There are two basic assumptions in developing SAE models, namely, that diversity in the response variable subpopulations can be explained entirely by the diversity of relations corresponding to the additional information, called a fixed effect, then the assumption of specific subpopulations diversity can be explained by a random effect subpopulation . A combination of both of these assumptions form a model of the effect of a mixture (mixed model). One of the interesting properties of linear mixed models is the ability to guess a linear combination of the effect of fixed and random effects. Henderson in 1963 (referred to [6]) developed a technique completion of a mixture of linear models, namely, Best Linear Prediction Unbiased Prediction (BLUP). This method is then studied further by Harville in 1991 (referred to [6]) by first estimation variance components with the maximum likelihood method (maximum likelihood) and constrained maximum likelihood (restricted maxim um likelihood), so-called Empirical Best Linear Unbiased Prediction (EBLUP).

A few years later EBLUP method developed by incorporating spatial effect into the model. EBLUP estimators by observing the effect of spatially correlated random area known as the Spatial Unbiased Empirical best linear prediction (SEBLUP) method. SEBLUP method can improve the structural diversity of small area estimation models that have a spatial correlation between areas. The model used in the method based SEBLUP area, the reason for modeling spatial incorporated into the model SAE is a spatial data type modeling area. SEBLUP estimators have been used by Petrucci and Salvati in [4] and Chandra, Salvati and Chambers in [1] to include spatial weighting matrix spatially nearest neighbors (nearest neighbors) into SEBLUP method.

The data obtained (variables of concern) usually have a large variance and tend to have a distribution pattern that is not symmetric and therefore caused no linear relationship pattern between concomitant variables and variables of concern. To resolve this problem, Kurnia [2] using the logarithmic transformation variables of concern are then applied to the methods and the results EBLUP estimation better than usual EBLUP methods. In this research, the logarithmic transformation variables of concern would be used in the method SEBLUP, which is expected to be obtained estimators with better precision.

PRELIMINARY THEORY

Direct Estimation

Implementation of the survey conducted to estimate population parameters. The classical approach to estimate population parameters based on design of sampling (design-based), and the estimation resulting from the approach is called direct estimation. Data from this survey can be used to obtain a reliable estimate of the total and the average population of an area or domain with a large number of examples. However, when the direct estimators are used to a small area, it will cause a large standard error (Ghosh and Rao in 1994, referred to [6]).

Indirect Estimation

In small area estimation, there are two basic types of models used, the basic area level model and the basic unit level models [6].

a) Basic area level models

A model based on the availability of supporting data that exists only for a particular area level, let $x_i = (x_{i1},\ x_{i2}, \dots, x_{ip})^T$ with parameters that will supposedly is θ_i are assumed to have a relationship with *xi*. The supporting data used to build the model $\theta_i = x_i{}^T\beta + z_i v_i$, with i = 1,2,3, ..., m. And $v_i \sim N(0, \sigma_v^2)$, as a random effect. Conclusions regarding θ_i, can be determined by assuming that the direct estimation models y_i already available, namely, $y_i = \theta_i + e_i$ with i = 1,2,3, ... m. and sampling error $e_i \sim N(0, \sigma_i^2)$, with σ_i^2 unknown. Then the two models are combined to obtain a combined models: : $y_i = x_i{}^T\beta + z_i v_i + e_i$, with i = 1,2,3,...m. The model is a special form of linear mixed models.

b) Basic unit level model

Is a model in which the supporting data provided corresponding individually with response data, such $x_{ij} = (x_{ij1},\ x_{ij2}, \dots, x_{ijp})^T$ to obtain a regression model nested $y_{ij} = x_{ij}{}^T\beta + v_i + e_{ij}$ with i = 1,2,3, ... m. and j = 1,2,3, ... Ni, $v_i \sim N(0, \sigma_v^2)$ and $e_i \sim N(0, \sigma_i^2)$.

Spatial Empirical Best Linear Unbiased Prediction (SEBLUP)

Suppose that defined the vector $\hat{\theta} = \left(\hat{\theta}_1, \dots, \hat{\theta}_m\right)^T$, $v = (v_1, \dots, v_m)^T$ and $e = (e_1, \dots, e_m)^T$, and the matrix $\boldsymbol{X} = (x_1^T, \dots, x_m^T)^T$ and **Z=** diag $(z_1, \dots, z_m)$ Based on the definition of the vectors and matrices, the equation (1) in matrix notation is:

$$\hat{\theta} = \boldsymbol{X\beta} + \boldsymbol{Zv} + \boldsymbol{e} \quad (1)$$

The model in equation (1) assumes that there is random effect area, these effects are independent between areas. In fact, it is reasonable to say that there is a correlation between the adjacent area. Such correlations will decrease as the distance increases. This is in accordance with the first law of geography that put forward by Tobler (*Tobler's first low of geography)* in [8] which is a pillar study spatial data analysis, that "*everything is related to everything else, but near things are more related then distant things "*. SAE models to include the spatial correlation between

areas was first introduced by Cressie in 1991 (referred to [6]), assuming that the spatial dependence follow Conditional Autoregressive process (autoregressive conditional, CAR). SAE models are then developed further by several researchers, including Petrucci and Salvati in [4], Chandra, Salvati and Chambers in [1], and Pratesi and Salvati in [5], assuming that the spatial dependency component that is inserted into the error of random factors follow the process Simultaneous Autoregressive (Simultaneous otoregresif, SAR). Model SAR was first introduced by Anselin in 1992 (referred to [1]) in which the random effect area vectors $\boldsymbol{v}$ satisfy area:

$$\boldsymbol{v} = \rho \boldsymbol{W}\boldsymbol{v} + \boldsymbol{u} \tag{2}$$

coefficient ρ in equation (2) is a spatial otoregresi coefficient indicates the strength of the relationship between the spatial random effect. The value of ρ ranging from -1 to 1. The value ρ> 0 indicates that an area with a high parameter values tend to be surrounded by other areas with high parameter values as well and an area with a low parameter value anyway. On the other hand, ρ <0 indicates that an area with a high parameter values are surrounded by other areas with a low parameter value, or vice versa [7]. **W** is a spatial weighting matrix, $\boldsymbol{v}$ is the random effect of the area and $\boldsymbol{u}$ is the error vector of random effects of an area with an average of zero and variance $\sigma_u^2 \boldsymbol{I}_{\mathrm{m}}$ Equation (2) can be rewritten as follows:

$$\boldsymbol{v} = (\mathbf{I} - \boldsymbol{v}\mathbf{W})^{-1}\boldsymbol{u} \tag{3}$$

Where **I** is the identity matrix with size $m \times m$. From equation (3) shows that the average $\boldsymbol{v}$ is 0 and the covariance matrix $\boldsymbol{v}$ **(G)** is as follows:

$$\mathbf{G} = \sigma_u^2 [(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^{\mathbf{T}})]^{-1}$$

Equation (3) is inserted into the equation (1) yields:

$$\widehat{\boldsymbol{\theta}} = \boldsymbol{X}\boldsymbol{\beta} + \mathbf{Z}(\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{u} + \boldsymbol{e}$$

Covariance matrix of $\widehat{\boldsymbol{\theta}}$ where **R** = diag (σ_i^2) :

$$\mathbf{V} = \mathbf{R} + \mathbf{Z}\mathbf{G}\mathbf{Z}^{\boldsymbol{T}} = \text{diag}\,(\sigma_i^2) + \mathbf{Z}\boldsymbol{\sigma}_{\boldsymbol{u}}^{\mathbf{2}}[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^{T})]^{-1}\boldsymbol{Z}^{T} \tag{4}$$

Spatial BLUP estimator for parameter θ_i with σ_u^2 , σ_i^2 and ρ known are:

$$\tilde{\theta}_i^S(\sigma_u^2, \rho) = x_i\hat{\beta} + \mathbf{b}_{\mathrm{i}}^{\mathrm{T}}\{\sigma_u^2 (\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^{\mathrm{T}})^{-1}\}\mathbf{Z}^{\mathrm{T}} \times\{\text{diag}\,(\sigma_i^2) + \mathbf{Z}\boldsymbol{\sigma}_{\boldsymbol{u}}^{\mathbf{2}}[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^{T})]^{-1}\boldsymbol{Z}^{T}\}^{-1}\big(\hat{\theta} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\big) \tag{5}$$

Where $\hat{\beta} = (\boldsymbol{X}^T V^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T V^{-1}\hat{\theta}$ and $\mathbf{b}_{\mathrm{i}}^{\mathrm{T}}$ is a vector of size $1 \times n$ $(0, 0, \ldots 0,\ 1,\ 0, \ldots .0)$ with 1 pointing to the location of the i-th. Spatial BLUP estimator is obtained by inserting covarians matrix in equation (4) into the BLUP estimator. Spatial BLUP will be the same as if ρ = 0. It can be concluded that estimators of small area estimation models to include the spatial correlation will lead to ordinary small area estimation model (random effects are independent) if in fact there is no spatial correlation in the area of the observed area , However, if there is a spatial correlation and if we use the model of the usual small area estimation models were also less precise. So, using the usual small area estimation models in areas that have a spatial correlation will produce a variety of structures that do not fit, when the model chosen less precise it will generate greater error.
MSE calculation of Spatial BLUP may be obtained as in Rao [6], namely:

$$MSE\big[\tilde{\theta}_i^S(\sigma_{\boldsymbol{u}}^{\mathbf{2}}, \rho)\big] = g_{1i}(\sigma_{\boldsymbol{u}}^{\mathbf{2}}, \rho) + g_{2i}(\sigma_{\boldsymbol{u}}^{\mathbf{2}}, \rho) \tag{6}$$

Where $g_{1i}(\sigma_{\boldsymbol{u}}^{\mathbf{2}}, \rho)$ and $g_{2i}(\sigma_{\boldsymbol{u}}^{\mathbf{2}}, \rho)$ are as follows:

$$g_{1i}(\hat{\sigma}_u^2, \rho) = \boldsymbol{b}_i^T\{\sigma_u^2 (\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)^{-1} - \sigma_u^2 (\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)^{-1}\boldsymbol{Z}^T \times \{\text{diag}\,(\sigma_i^2) + \mathbf{Z}\boldsymbol{\sigma}_{\boldsymbol{u}}^{\mathbf{2}}[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T\}^{-1}\boldsymbol{Z}\sigma_u^2 \times [(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\}$$

$$\begin{aligned} g_{2i}(\hat{\sigma}_u^2, \rho) = {} & \boldsymbol{x_i} - \boldsymbol{b}_i^T\sigma_u^2 [(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T \times \{\text{diag}\,(\sigma_i^2) + \mathbf{Z}\boldsymbol{\sigma}_{\boldsymbol{u}}^{\mathbf{2}}[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T\}^{-1}\boldsymbol{X} \\ & \times (\boldsymbol{X}^{\boldsymbol{T}}\{\text{diag}\,(\sigma_i^2) + \mathbf{Z}\boldsymbol{\sigma}_{\boldsymbol{u}}^{\mathbf{2}}[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T\}^{-1}\boldsymbol{X})^{-1} \times \boldsymbol{x_i} \\ & - \boldsymbol{b}_i^T\sigma_u^2 [(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T \\ & \times (\{\text{diag}\,(\sigma_i^2) + \mathbf{Z}\boldsymbol{\sigma}_{\boldsymbol{u}}^{\mathbf{2}}[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T\}^{-1}\boldsymbol{X})^{-1} \end{aligned}$$

As is the case with EBLUP estimator ,SEBLUP estimator $\left(\hat{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})\right)$ from Spatial BLUP obtained by replacing the value σ_u^2, ρ with the estimators. The assumption of normality of random effects is used to predict σ_u^2 dan ρ using both ML and REML procedure with the log-likelihood function has a global maximum and several local maximum [1, 5]. The probe can be obtained iteratively by using a scoring algorithm. Scoring algorithm requires a great starting point to get the maximum functionality. The estimation results are then used to probe against SEBLUP, with estimators EBLUP formula is:

$$\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho}) = x_i\hat{\beta} + \boldsymbol{b}_i^T\{\hat{\sigma}_u^2 (\boldsymbol{I} - \hat{\rho}\boldsymbol{W})(\boldsymbol{I} - \hat{\rho}\boldsymbol{W}^T)^{-1}\}\boldsymbol{Z}^T$$
$$\times\{\text{diag}(\sigma_i^2) + \mathbf{Z}\hat{\sigma}_u^2 [(\mathbf{I} - \hat{\rho}\mathbf{W})(\mathbf{I} - \hat{\rho}\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T\}^{-1}\left(\hat{\theta} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\right) \quad (7)$$

$MSE\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho})\right]$ EBLUP spatial models with random effects are normally distributed, are:

$$MSE\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho})\right] = MSE\left[\tilde{\theta}_i^S, (\boldsymbol{\sigma_u^2}, \rho)\right] + E\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho}) - \tilde{\theta}_i^S, (\boldsymbol{\sigma_u^2}, \rho)\right]^2 \quad (8)$$

Form $E\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho}) - \tilde{\theta}_i^S, (\boldsymbol{\sigma_u^2}, \rho)\right]^2$ estimated by Taylor and denoted by $g_{3i}(\sigma_u^2, \rho)$ (Kackar and Harville in 1984, referred to [5]) ,that is:

$$g_{3i}(\sigma_u^2, \rho) = tr\left\{\begin{bmatrix} \boldsymbol{b_i^T}\left(\boldsymbol{C^{-1}Z^TV^{-1}} + \sigma_u^2\boldsymbol{C^{-1}Z^T}(-\boldsymbol{V^{-1}ZC^{-1}Z^TV^{-1}})\right) \\ \boldsymbol{b_i^T AZ^TV^{-1}} + \sigma_u^2\boldsymbol{C^{-1}Z^T}(-\boldsymbol{V^{-1}ZAZ^TV^{-1}}) \end{bmatrix}\boldsymbol{V}\right.$$
$$\left.\times\begin{bmatrix} \boldsymbol{b_i^T}\left(\boldsymbol{C^{-1}Z^TV^{-1}} + \boldsymbol{\sigma_u^2}C^{-1}\boldsymbol{Z^T}(-\boldsymbol{V^{-1}ZC^{-1}Z^TV^{-1}})\right) \\ \boldsymbol{b_i^T}\left(\boldsymbol{AZ^TV^{-1}} + \boldsymbol{\sigma_u^2C^{-1}Z^T}(-\boldsymbol{V^{-1}ZAZ^TV^{-1}})\right) \end{bmatrix}^T \overline{\boldsymbol{V}}(\hat{\sigma}_u^2, \hat{\rho})\right\}$$

With $\boldsymbol{C} = [(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]$, $\boldsymbol{A} = -\sigma_u^2[\mathbf{C^{-1}}(2\rho\mathbf{WW^T} - \mathbf{2W})\mathbf{C^{-1}}]$ and $\overline{\boldsymbol{V}}(\hat{\sigma}_u^2, \hat{\rho})$ the asymptotic covariance matrix of $\hat{\sigma}_u^2$ and $\hat{\rho}$.

Estimators of $MSE\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho})\right]$ obtained by following the results of Harville and Jeske in 1992 (referred to [5]) and later developed into a model with generalized covariances by Zimmerman and Cressie in 1992 (referred to [5]), namely

$$mse\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho})\right] \approx g_{1i}(\hat{\sigma}_u^2, \hat{\rho}) + g_{2i}(\hat{\sigma}_u^2, \hat{\rho}) + 2g_{3i}(\hat{\sigma}_u^2, \hat{\rho})$$

Where $\hat{\sigma}_v^2$ and $\hat{\rho}$ is the estimator obtained using REML method. If using $\hat{\sigma}_v^2$ and $\hat{\rho}$ ML estimators procedures, calculations $mse\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho})\right]$ as follows: $mse\left[\tilde{\theta}_i^S, (\hat{\sigma}_u^2, \hat{\rho})\right] \approx$
$g_{1i}(\hat{\sigma}_u^2, \hat{\rho}) - b_{ML}^T(\hat{\sigma}_u^2, \hat{\rho})\nabla g_{1i}(\hat{\sigma}_u^2, \hat{\rho}) + g_{2i}(\hat{\sigma}_u^2, \hat{\rho}) + 2g_{3i}(\hat{\sigma}_u^2, \hat{\rho})$
with $b_{ML}^T(\hat{\sigma}_v^2, \hat{\rho})\nabla g_{1i}(\hat{\sigma}_v^2, \hat{\rho})$ obtained as follows:

$\nabla g_{1i}(\hat{\sigma}_u^2, \hat{\rho})$

$$= \boldsymbol{b_i^T}\left\{\begin{matrix}(\boldsymbol{C^{-1}} - [\boldsymbol{C^{-1}Z^TV^{-1}Z}\sigma_u^2\boldsymbol{C^{-1}} + \sigma_u^2\boldsymbol{C^{-1}Z^T}(-\boldsymbol{V^{-1}ZC^{-1}Z^TV^{-1}})\boldsymbol{Z}\sigma_u^2\boldsymbol{C^{-1}} + \sigma_u^2\boldsymbol{C^{-1}Z^TV^{-1}ZC^{-1}}]) \\ (\boldsymbol{A} - [\boldsymbol{AZ^T}V^{-1}\boldsymbol{Z}\sigma_u^2\boldsymbol{C^{-1}} + \sigma_u^2\boldsymbol{C^{-1}Z^T}(-\boldsymbol{V^{-1}ZAZ^TV^{-1}})\boldsymbol{Z}\sigma_u^2\boldsymbol{C^{-1}} + \sigma_u^2\boldsymbol{C^{-1}Z^TV^{-1}ZA}])\end{matrix}\right\}\boldsymbol{b_i}$$

and

$$b_{ML}^T(\sigma_u^2, \rho) = \frac{1}{2m}\left\{\beth^{-1}(\sigma_u^2, \rho)\begin{bmatrix} tr[(\boldsymbol{X^TV^{-1}X})^{-1}\boldsymbol{X^T}(-\boldsymbol{V^{-1}ZC^{-1}Z^TV^{-1}})\boldsymbol{X}] \\ tr[(\boldsymbol{X^TV^{-1}X})^{-1}\boldsymbol{X^T}(-\boldsymbol{V^{-1}ZAZ^TV^{-1}})\boldsymbol{X}] \end{bmatrix}\right\}$$

$b_{ML}^T(\hat{\sigma}_u^2, \hat{\rho})\nabla g_{1i}(\hat{\sigma}_u^2, \hat{\rho})$ form is an additional form of an additional bias of $g_{1i}(\hat{\sigma}_u^2, \hat{\rho})$.

Logarithmic Transformation

Defined a logarithmic transformation in linear mixed models (log-scale linear mixed model) as follows:

$$\log(y_{ij}) = \boldsymbol{x_{ij}}^T \boldsymbol{\beta} + v_i + \varepsilon_{ij} \quad (9)$$

with ε_{ij} following the distribution iid N(0, σ_i^2), random effect of area v_i following the distribution iid N(0, σ_v^2) but if there is a spatial effect of the v_i following the distribution MVN (0, G). Kurnia in [2] explained that following EBLUP theory to model (9), namely EBLUP to the mean θ_i of $\log(y_{ij})$, then estimators for θ_i can be written as follows:

$$\hat{\theta}_i^{EBLUP*} = \hat{\gamma}_i \hat{\theta}_i^D + (1 - \hat{\gamma}_i)\, \boldsymbol{x_i}^T \widehat{\boldsymbol{\beta}} \quad (10)$$

with $\hat{\beta}$ obtained by weighted least squares method for parameter β regression of log-scale linear mixed model, where $\hat{\gamma}_i = \hat{\sigma}_v^2/(\hat{\sigma}_v^2 + n_i^{-1}\hat{\sigma}_i^2)$ and, $\hat{\theta}_i^D = \frac{1}{n_i}\sum_{j\in s(i)} \log(y_{ij})$ is a direct estimator for θ_i based on data sample s (i) for the i-th area.

Because we want is an actual estimate for the median in each area to-i, then used a lognormal distribution properties to transform and forth from the model (10). Furthermore, it is assumed that the sampling distribution for $\hat{\theta}_i^{EBLUP}$ is $N\{\theta_i, Var(\hat{\theta}_i^{EBLUP*})\}$. Thus, actual estimate value (raw-scale) for the i-th area is

$$\hat{\theta}_i^{LOG\ SCALE\ EBLUP} = exp\left(\hat{\theta}_i^{EBLUP*} + \frac{1}{2}\hat{v}_i^{EBLUP*}\right) \quad (11)$$

With $\hat{v}_i^{EBLUP*}$ is MSE estimators of $\hat{\theta}_i^{EBLUP*}$. Then the MSE estimator for the mean estimator in equation (11) can be obtained as follows:

$$\hat{V}_i\left(\hat{\theta}_i^{LOG\ SCALE\ EBLUP}\right) = e^{\hat{v}_i^{EBLUP*}}\left(e^{\hat{v}_i^{EBLUP*}} - 1\right) e^{2\hat{\theta}_i^{EBLUP*}} \quad (12)$$

In this study, will be applicable log-scale linear mixed models into SEBLUP method. by following SEBLUP theory to model (9), namely SEBLUP to the mean θ_i of $\log(y_{ij})$, then estimator for θ_i can be written as follows:

$$\hat{\theta}_i^{SEBLUP*} = \boldsymbol{x_i}\widehat{\boldsymbol{\beta}} + \boldsymbol{b}_i^T\{\hat{\sigma}_u^2\,(\boldsymbol{I} - \hat{\rho}\boldsymbol{W})(\boldsymbol{I} - \hat{\rho}\boldsymbol{W}^T)^{-1}\}\boldsymbol{Z}^T$$
$$\times\{\text{diag}\,(\hat{\sigma}_i^2) + \mathbf{Z}\hat{\sigma}_u^2\,[(\mathbf{I} - \hat{\rho}\mathbf{W})(\mathbf{I} - \hat{\rho}\mathbf{W}^T)]^{-1}\boldsymbol{Z}^T\}^{-1}\left(\widehat{\boldsymbol{\theta}}^{\boldsymbol{D}} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\right) \quad (13)$$

With, $(\widehat{\boldsymbol{\theta}}^{\boldsymbol{D}})^T$: $(\hat{\theta}_1^D, \hat{\theta}_2^D, \hat{\theta}_3^D, \ldots\ldots \hat{\theta}_m^D)$ and $\hat{\theta}_i^D = \frac{1}{n_i}\sum_{j\in s(i)} \log(y_{ij})$.). as well as EBLUP, the method is desirable also SEBLUP actual estimator for the mean in each area to-i, thus obtained:

$$\hat{\theta}_i^{LOG\ SCALE\ SEBLUP} = exp\left(\hat{\theta}_i^{SEBLUP*} + \frac{1}{2}\hat{v}_i^{SEBLUP*}\right) \quad (14)$$

With $\hat{v}_i^{SEBLUP*}$ is MSE estimators of dari $\hat{\theta}_i^{SEBLUP*}$. Then the MSE estimator for mean estimation in the equation (14) can be obtained as follows:

$$\hat{V}_i\left(\hat{\theta}_i^{LOG\ SCALE\ SEBLUP}\right) = e^{\hat{v}_i^{SEBLUP*}}\left(e^{\hat{v}_i^{SEBLUP*}} - 1\right) e^{2\hat{\theta}_i^{SEBLUP*}} \quad (15)$$

METHOD

Simulation Study

Simulations were performed to evaluate the good of the developed model. The simulation process is done by following these steps.

1. Make a map made in the form of a grid consisting of a small area m. Where m is to be tested are 49
2. Build a sample consisting of m small area
3. search for spatial contiguity weighting matrix Queen (W) based on a map that has been created.
4. This simulation uses the response variable Y(variable of concern) and an accompanying variable X. The model used to derive the value of the response variable log(y_{ij}) for small area i-th and j-th unit is as follows:

$$\log\left(y_{ij}\right) = \beta_0 + \beta_1 x_{ij} + v_i + e_{ij} \quad ,i = 1,2,\ldots,49, \qquad j = 1,2,\ldots,n_i \quad (16)$$

Where x_{ij} is concomitant variables, v_i is random effect area, and e_{ij} is sampling error.

(a) x_{ij} value generated by a normal distribution $N(2,1)$. x_{ij} value obtained is used for the entire scenario in the simulation process.
(b) then $\boldsymbol{v} = (v_1, \dots, v_m)^T$ raised by the spread Multivariat Normal MVN (**0, G**), where $\mathbf{G}=\sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}$ is the variance-covariance matrix size 49 × 49 . determined value $\sigma_u^2 = 0.5, 1, 2, 3$ and $\rho = 0.25,\ 0.5,\ 0.75$
(c) Then, $\mathbf{e} = (e_{11},\ e_{12}, \dots, e_{ij}, \dots, e_{mN_m})^T$ generated by a normal distribution $N(0, 0.34)$.
(d) Last value, set $\boldsymbol{\beta} = (1,1)^T$ in order to obtain the following equation:

$$\log(y_{ij}) = 1 + x_{ij} + v_i + e_{ij} \quad , i = 1,2, \dots ,49,\ \ j = 1,2, \dots, n_i. \qquad (17)$$

(e) Determine the value y_{ij} by entering the value x_{ij}, v_i and e_{ij} into the model or equation (17)
(f) Looking for the actual value y_{ij} with $y_{ij} = e^{\log(y_{ij})}$, so it can be said y_{ij} generated with a log-normal distributon.
(g) Find the parameter :

$$\theta_i = e^{(\boldsymbol{x_i}^T\boldsymbol{\beta}+v_i)+\frac{1}{2}\sigma_e^2}$$

5. Calculating the mean variable of concern samples in each small area as a direct estimation
   $\hat{\theta}_i = \hat{\bar{y}}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$, with $i = 1,2, \dots ,49 \qquad j = 1,2, \dots, n_i$
   Then calculate the mean variable concomitant each sample in a small area
   $\boldsymbol{x_i} = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}$, with $i = 1,2, \dots ,49 \qquad j = 1,2, \dots, n_i$
   Calculating the mean logarithmic variable of concern samples in each small area as a direct estimation
   $\hat{\theta}_i^D = \hat{\bar{y}}_i^{\ D} = \frac{1}{n_i}\sum_{j=1}^{n_i} \log(y_{ij})$, with $i = 1,2, \dots ,49, \qquad j = 1,2, \dots, n_i$
6. Find the value of $\hat{\theta}_i^{EBLUP}$ using $\hat{\theta}_i$
7. Find the value of $\hat{\theta}_i^{SEBLUP}$ using $\hat{\theta}_i$.
8. Find the value of $\hat{\theta}_i^{LOG\ SCALE\ EBLUP}$ using $\hat{\theta}_i^D$. After that is done, behind the transformation.
9. Find the value of $\hat{\theta}_i^{LOG\ SCALE\ SEBLUP}$ using $\hat{\theta}_i^D$. After that is done, behind the transformation.
10. Perform steps (4) to step (8) of B = 1000 so it can be calculated the value of the Relative Bias (RB) and Relative Root Mean Squares Error (RRMSE) of parameter estimation results in each area as follows:

$$RB_{(i)} = \frac{1}{B}\sum_{l=1}^{B}\left(\frac{\hat{\theta}_{il} - \theta_i}{\theta_i}\right) \times 100\%$$

$$RRMSE_{(i)} = \frac{1}{\theta_i}\sqrt{\frac{1}{B}\sum_{l=1}^{B}\left(\hat{\theta}_{il} - \theta_i\right)^2} \times 100\%$$

$$\overline{ARB} = \frac{1}{m}\sum_{i=1}^{m} ARB_{(i)}$$

$$\overline{RRMSE} = \frac{1}{m}\sum_{i=1}^{m} RRMSE_{(i)}$$

11. Compare the value of $\overline{ARB}$ and $\overline{RRMSE}$ between the EBLUP methods, EBLUP methods, LOG SCALE EBLUP methods, LOG SCALE SEBLUP methods

RESULTS AND DISCUSSION

Study of the simulation was conducted with four estimators, namely: (1) EBLUP using $\hat{\theta}_i$ (EBLUP), (2) SEBLUP using $\hat{\theta}_i$ (SEBLUP), (3) back transformation of EBLUP using $\hat{\theta}_i^D$ (LOG SCALE EBLUP) , (4) back transformation of SEBLUP using $\hat{\theta}_i^D$ (LOG SCALE SEBLUP) and while the results are as follows:

TABLE 1. Simulation Study Results Summary

u_i		ρ=0,75			
		EBLUP	**SEBLUP**	**LOG SCALE EBLUP**	**LOG SCALE SEBLUP**
N(0 , 0.5)	$\overline{\text{RRMSE}}$(%)	62,48	62,57	15,67	15,61
	$\overline{\text{ARB}}$ (%)	61,38	61,55	15,26	15,20
N(0 , 1)	$\overline{\text{RRMSE}}$(%)	62,70	62,76	15,67	15,63
	$\overline{\text{ARB}}$ (%)	61,38	61,53	15,26	15,22
N(0 , 2)	$\overline{\text{RRMSE}}$(%)	62,97	62,97	15,68	15,65
	$\overline{\text{ARB}}$ (%)	61,63	61,61	15,26	15,23
N(0 , 3)	$\overline{\text{RRMSE}}$(%)	63,23	63,21	15,68	15,65
	$\overline{\text{ARB}}$ (%)	61,86	61,84	15,26	15,23
u_i		ρ=0,5			
		EBLUP	**SEBLUP**	**LOG SCALE EBLUP**	**LOG SCALE SEBLUP**
N(0 , 0.5)	$\overline{\text{RRMSE}}$(%)	62,41	62,48	15,65	15,62
	$\overline{\text{ARB}}$ (%)	61,38	61,49	15,24	15,21
N(0 , 1)	$\overline{\text{RRMSE}}$(%)	62,60	62,65	15,66	15,64
	$\overline{\text{ARB}}$ (%)	61,43	61,54	15,25	15,22
N(0 , 2)	$\overline{\text{RRMSE}}$(%)	62,81	62,80	15,67	15,65
	$\overline{\text{ARB}}$ (%)	61,37	61,47	15,26	15,23
N(0 , 3)	$\overline{\text{RRMSE}}$(%)	62,96	62,92	15,67	15,66
	$\overline{\text{ARB}}$ (%)	61,58	61,53	15,26	15,24
u_i		ρ=0,25			
		EBLUP	**SEBLUP**	**LOG SCALE EBLUP**	**LOG SCALE SEBLUP**
N(0 , 0.5)	$\overline{\text{RRMSE}}$(%)	62,38	62,44	15,63	15,62
	$\overline{\text{ARB}}$ (%)	61,39	61,46	15,23	15,21
N(0 , 1)	$\overline{\text{RRMSE}}$(%)	62,57	62,61	15,65	15,64
	$\overline{\text{ARB}}$ (%)	61,39	61,53	15,24	15,23
N(0 , 2)	$\overline{\text{RRMSE}}$(%)	62,73	62,74	15,66	15,66
	$\overline{\text{ARB}}$ (%)	61,45	61,51	15,25	15,24
N(0 , 3)	$\overline{\text{RRMSE}}$(%)	62,84	61,83	15,67	15,66
	$\overline{\text{ARB}}$ (%)	61,40	61,46	15,25	15,24

From Table 1 it can be seen that the $\overline{\text{ARB}}$ (%) value for the LOG SCALE EBLUP methods and LOG SCALE SEBLUP methods always smaller than the EBLUP methods and SEBLUP methods, this indicates that the data has a distribution curve that is not symmetric would be better if logarithmic transformation is done first and then inserted into a small area estimation methods that will result in a smaller bias compared with the method without transformation first. From table 1 can also be seen that $\overline{\text{ARB}}$ (%) for the LOG SCALE SEBLUP methods almost every scenario has a smaller value if compared with LOG SCALE EBLUP methods, but the vagaries of this simulation is the $\overline{\text{ARB}}$ (%) value of SEBLUP methods is always greater compared with EBLUP methods, whereas the data generated in this simulation with spatial effect that can be seen from, $v_i = \left(I - \rho W\right)^{-1} u_i$. although v_i was raised with $\rho = 0.75$ (spatial effect between area high), still produces a $\overline{\text{ARB}}$ (%) value greater for SEBLUP method instead of the EBLUP. This indicates that the data has a spatial effect and have a distribution that is not symmetrical, it is not enough if handled with the usual methods such as SEBLUP methods because it will produce biased estimator with sizeable. So that the most appropriate method in this case is the LOG SCALE SEBLUP

methods which has the smallest bias instead of the LOG SCALE EBLUP methods, SEBLUP methods and EBLUP methods. Although, in fact LOG SCALE SEBLUP methods is biased and not known what causes it.

From Table 1 it can be seen that the $\overline{\text{RRMSE}}$ (%) value for the LOG SCALE EBLUP methods and LOG SCALE SEBLUP methods always smaller than the EBLUP methods and SEBLUP methods, this indicates that the data has a distribution curve that is not symmetric would be better if logarithmic transformation is done first and then inserted into a small area estimation methods that will produce a smaller error compared to the method without transformation first. From table 1 can also be seen that $\overline{\text{RRMSE}}$ (%) for the LOG SCALE SEBLUP methods almost every scenario has a smaller value if compared with LOG SCALE EBLUP methods, and the $\overline{\text{RRMSE}}$ (%) value of the SEBLUP methods always greater compared with EBLUP methods, for $v_i = \left(I - \rho W\right)^{-1} u_i$ where $\sigma_u^2 = 0.5,\ 1,\ 2$. But for $\sigma_u^2 = 3$ then the $\overline{\text{RRMSE}}$ (%)value of SEBLUP methods smaller than the EBLUP methods. So changes σ_u^2 greater effect on the $\overline{\text{RRMSE}}$ (%)value. But in general, the $\overline{\text{RRMSE}}$ (%)value for the LOG SCALE SEBLUP methods always smaller than the LOG SCALE EBLUP methods, SEBLUP methods and EBLUP methods. LOG SCALE SEBLUP methods so the method can be said that the results were better estimation seen from $\overline{\text{RRMSE}}$ smaller.

CONCLUSIONS

Estimation of small area which has a spatial effect is usually done by using SEBLUP (Spatial Best Linear Unbiased Prediction), while the variables of concern usually have a symmetric distribution so that the pattern of relationship variables of concern and concomitant variables be nonlinear. So we need a method of estimation that can address these problems and the method is LOG SCALE SEBLUP which is a method to perform logarithmic transformation of the variables of concern then applied to the SEBLUP methods and after that is done back transformation. Results of the simulation study showed that the LOG SCALE SEBLUP methods have $\overline{\text{ARB}}$ value smaller in each scenario if it is compared with other methods but the estimation LOG SCALE SEBLUP methods still have a biased probe and unknown causes. Then, for the $\overline{\text{RRMSE}}$ value, LOG SCALE SEBLUP method has a smaller value when compared to other methods. So in this case, the LOG SCALE SEBLUP methods able to improve efficiency of estimation.

REFERENCES

1. Chandra H, Salvati N, Chambers R., Small Area Estiamtion for Spatially correlated populations a comparison of direct and indirect model-based methodes, *Statistics in transition* 8:887-906, 2007.
2. Kurnia A., *An Empirical Best Prediction Method for Logarithmic Transformation Model in Small Area Estimation with Particular Application to Susenas Data*, Master Student, Bogor Agricultular University, 2009.
3. Petrucci A, Salvati N., Small area estimation using spatial information, The rathbun lake watershed case study, *Dipartimento di Statistica "G.Parenti" viale morgagani*, 59-50134, 2004a.
4. Petrucci A, Salvati N., Small area estimation considering spatially errors : the unit level random effects model. *Dipartimento di Statistica "G.Parenti" viale morgagani*, 59-50134, 2004b.
5. Pratesi M, Salvati N., Small Area Estimation: the EBLUP estimator based on spatially correated random area effects. Statistical methodes and applications, Stat.Meth. & Appl. 17:113-141, 2008.
6. Rao JNK., *Small Area Estiamtion*. London : Wiley, 2003.
7. Savitz NV, Raudenbush SW., Exploiting spatial dependence to improve measurement of neighborhood social processes, *Sociological Methodology*, 39: 151, 2009.
8. Schabenberger O, Gotway CA., Statistical Methodes for Spatial Data Anlysis, Chapman & Hall /CRC, 2005.

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 75–83.

Multivariable Additive Nonparametric Regression Curve Estimations To Use Non-Trend Fourier Series Estimators

I Wayan Sudiarsa[1], I Nyoman Budiantara[2], Suhartono[3],Santi Wulan Purnami[4]

[1,2,3,4] Statistics Departement, FMIPA, Institut Teknologi Sepuluh November (ITS)
JL. Arief Rahman Hakim, Surabaya 60111 Indonesia
[1]wsudiarsa72@yahoo.com

Abstract. Multivariable additive nonparametric regression model is a nonparametric regression model that involves more than one predictor and has additively separable function on each predictor. There are many functions that can be used on nonparametric regression models, such as the kernel, splines, wavelets, lokal polinomial and fourier series. This research focuses on multivariable additive nonparametric regression models a result of between non trend fourier series. The estimation method that be used to obtain the estimators is Penalized Least Square. This method requires the estimation of smoothing parameters in the optimation process for obtaining the estimator of multivariable additive nonparametric regression model has been succesfully obtained, which consits an estimator of fourier series non-trend. The results of this theoritical study shows that the Penalized Least Square method works simultan coessly for obtaining the estimators of the smoothing parameters an nonparametric regression models parameters as a result of non-trend fourier series which are additively separable.

Keywords: Multivariable nonparametric regression, Fourier Series, Non-Trend, Penalized Least Square.

INTRODUCTION

Parametric regression is the method most often used to complete the modeling of the relationship between response and predictor variables when the shape of the functional relationship is known or assumed to follow a specific function. In practice often parametric regression is not conform to represent the relationship between response and predictor variables caused no known form of the functional relationship of both variables. In the last decade has much to do studies on nonparametric regression is used to resolve the relationship between response and predictor variables when information is not known functional relationship. Along with the development of computing and some limitations on the parametric regression model nonparametric regression requires many assumptions that have become more widely applied to solve problems in various fields of applied [10] .

Some research that associated with estimator in nonparametric regression model can be seen in [11], [14], [20], [8] for the kernel estimator. [4], [9], as well as [2] for the spline estimator. [17], [7], as well as [13] for local polynomial estimator. [15], [16], and [18] for the wavelet estimators. [1], [5], [12] and [6] for the Fourier series estimator.

Recent developments relating to research on nonparametric regression model showed that the Fourier series is one of the alternative estimator that end - the end is much studied and developed by researchers nonparametric regression. [3] is a group of early researchers who study about the transformation of Fourier series for smoothing function of density, especially in smoothing the spectral estimator. In 1992, Bilodeau examines the Fourier series estimator in nonparametric regression with additive and component predictor Penalized least squares to get the coefficient - coefficient. Fourier series estimator used is the sum of a linear function and a trigonometric polynomial functions.

Between the models of the nonparametric regression, Fourier series is one of the models that have a statistical and visual interpretation of a very special and nice [19]. Estimation of Fourier series is capable of handling data that is smooth character and follow the pattern repeated at certain intervals [1]. Fourier series that was developed in the nonparametric regression is a Fourier series with the predictor variables have an element

of trend up or down trend. On this trend shown by adding one tribe linear equations in Fourier series model. Fourier series estimator developed by the trend was generally very appropriate for the data pattern repeated at certain intervals and follow the trend up or down trend [1, 16]. Problem will occur in the use of Fourier series estimator if there is a data pattern repeated at certain intervals, but do not have the rising trend and the trend down. It is therefore necessary

to develop a Fourier series estimator that does not contain the elements of the trend curve estimating nonparametric regression multivariable additive.

FORMULATION PROBLEM

How to shape nontrend Fourier series estimator in nonparametric regression curve to estimate multivariable additive.

PURPOSES OF RESEARCH

Getting the form of Fourier series estimator nontrend to estimate multivariable additive nonparametric regression curve.

LITERATURE RIVIEW

Fourier series nontrend Estimator to estimate multivariable nonparametric regression curve.

Fourier series estimator with multivariable trend, in recent years a lot of attention from several investigators nonparametric regression. Generally used when the data pattern is not known to be investigated and there is a tendency seasonal pattern (Tripena and Budiantara, 2007; Bilodeau, 1992).

Lets univariable nonparametric regression model $y_i = g(x_i) + \varepsilon_i$, i = 1, 2, 3, ..., n. The from of curve regression $g(x)$ is assumed unknown and contained in a continuous function space. $C(0,\pi)$. Random error ε_i is assumed to be normally distributed with the independent variant. (σ^2). because $g(x)$ continuous on the interval $(0,\pi)$, it can be approached by Fourier series with trend function $F(x)$ with:

$$F(x) = bx + \frac{1}{2}a_o + \sum_{k=1}^{K} a_k \cos kx$$

when $b, a_o, a_k, k = 1,2,...,K$ the parameters of model. Based on the Fourier series model. Bilodeau (1992) gives the Fourier series estimator with univariable following trend:

Teorema 1. [1] *If the nonparametric function univariable* $g(x)$ *approached by function* $F(x)$*, then the Fourier series estimator obtained with the trend of minimize:*

$$n^{-1}\sum_{j=1}^{n}\left(y_j + bx - \frac{1}{2}a_o - \sum_{k=1}^{K} a_k \cos kx \right) + w\sum_{k=1}^{K} i^2 a_i^2, w > 0$$

for each $b \in \Re,\ a_0 \in \Re,\ a_i \in \Re,\ i = 1,2,...,K$ *let by:*

$$\hat{F}_w(x) = \hat{b}(w)x + \frac{1}{2}\hat{a}_o(w) + \sum_{k=1}^{K} \hat{a}_k(w)\cos kx$$

with $\hat{\underset{\sim}{a}}(w) = \left(\hat{b}(w)x, \hat{a}_o(w), \hat{a}_1(w)..., \hat{a}_k(w)\right)'$ *is obtained from equation:*

$$\hat{\underset{\sim}{a}}(w) = \left(\hat{b}(w)x, \hat{a}_o(w), \hat{a}_1(w)..., \hat{a}_k(w)\right)'$$

when D diagonal matrix, let by $D = diag\left(0,1^4,2^4,...,K^4\right)$ *and x coefision matrix.*

RESEARCH METHOD

Obtaining Fourier series estimator nontrend to estimate multivariable nonparametric regression curve.

1. Let a multivariable nonparametric additive regression model:

$$y_i = g_1\left(x_{1i}\right) + g_2\left(x_{2i}\right) + ... + g_p\left(x_{pi}\right) + \varepsilon_i; i = 1,2,...,n$$

2. Approach regressions curve $g_1\left(x_1\right), g_2\left(x_2\right), ..., g_p\left(x_p\right)$ Fourier series functions nontrend that is:

$$g\left(x\right) = \frac{1}{2}a_o + \sum_{k=1}^{K} a_k \cos kx$$

3. Searching for the size of the goodness of fit for optimization PLS:

$$n^{-1}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p} g_j\left(x_{ji}\right)\right)^2$$

4. Searching the value of penalty for PLS:

$$\sum_{j=1}^{p}\lambda_j\int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2 dx_{ji}$$

5. Complete the optimization Penalized Least Squares (PLS):

$$\underset{g\in C(0,\pi)}{Min}\left\{n^{-1}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p} g_j\left(x_{ji}\right)\right)^2 + \sum_{j=1}^{p}\lambda_j\int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2 dx_{ji}\right\}$$

FORM OF ESTIMATOR FOURIER SERIES NONTREND IN NONPARAMETRIC REGRESSION CURVES ESTIMATING MULTIVARIABLE ADDITIVE

In this section will be presented nontrend Fourier series estimator in nonparametric regression multivariable additive, which is let by the equation:

$$\begin{aligned} y_i &= \mu\left(x_{1i}, x_{2i}, ..., x_{qi}\right) + \varepsilon_i \\ &= \sum_{j=1}^{q} g_j\left(x_{ji}\right) + \varepsilon_i, i = 1,2,...,n. \end{aligned} \quad (1)$$

The next regression curve f_j approached with the function of Fourier series nontrend:

$$g_j\left(x_{ji}\right) = \frac{1}{2}a_{oj} + \sum_{k=1}^{K} a_{kj}\cos kx_{ji}, j = 1,2,...,q \quad (2)$$

Fourier series regression curve estimation nontrend obtained from optimization:

$$\underset{B\in R^{q(k+j)}}{Min}\left\{n^{-1}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{q} g_j\left(x_{ji}\right)\right)^2\right\} \quad (3)$$

With the provision of:

$$J\left(g_j\right) = \int_0^{\pi}\frac{2}{\pi}\left\{g_j''\left(x_{ji}\right)\right\}^2 dx_{ji} \le \rho, \rho \ge 0 \quad (4)$$

Completion of optimization (3) with the condition (4) is equivalent to completing the optimization:

$$\underset{B\in R^{q(k+j)}}{Min}\left\{n^{-1}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{q} g_j\left(x_{ji}\right)\right)^2 + \sum_{j=1}^{q}\lambda_j\int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2 dx_{ji}\right\}. \quad (5)$$

Parameter λ Parameter is a smoothing parameter that measures the goodness of fit $R\left(g_j\right)$and smoothing $J\left(g_j\right)$.Fourier series estimator nontrend shape obtained from optimization (3) Let by some of lemma relating to proof of the theorem 1 as next:

Lemma 1. *If the given function nontrend fourier series* g_j *as in equation (6.1), then:*

$$J\left(g_j\right)=\int_0^{\pi}\frac{2}{\pi}\left(g_j''\left(x_{ji}\right)\right)^2 dx_{ji}=\sum_{k=1}^{K}k^4a_{jk}^2,\ j=1,2,...,r.$$

Proof: Because $g_j\left(x_j\right)=\frac{1}{2}a_o+\sum_{k=1}^{K}a_{kj}\cos kx_j$, then

$$g_j''\left(x_j\right)=\frac{d}{dx_j}\left(\frac{d}{dx_j}\left(\frac{1}{2}a_o+\sum_{k=1}^{K}a_{kj}\cos kx_j\right)\right)$$

$$=-\sum_{k=1}^{K}k^2a_{kj}\cos kx_j$$

$$\left(g_j''\left(x_j\right)\right)^2=\left(\sum_{k=1}^{K}k^2a_{kj}\cos kx_j\right)^2$$

Penalty $J\left(g_j\right)$ be a:

$$J\left(g_j\right)=\int_0^{\pi}\frac{2}{\pi}\left(\sum_{k=1}^{K}k^2a_{kj}\cos kx_j\right)^2 dx_j$$

$$=\frac{2}{\pi}\int_0^{\pi}\left(\sum_{k=1}^{K}k^2a_{kj}\cos kx_j\right)^2 dx_j+\frac{4}{\pi}\int_0^{\pi}\sum_{k<j}^{K}\left(k^2a_{kj}\cos kx_j\right)\left(j^2a_j\cos jx_j\right)dx_j$$

Suppose:

$$A=\frac{2}{\pi}\int_0^{\pi}\left(\sum_{k=1}^{K}k^2a_{kj}\cos kx_j\right)^2 dx_j \text{ and } B=\frac{4}{\pi}\int_0^{\pi}\sum_{k<j}^{K}\left(k^2a_{kj}\cos kx_j\right)\left(j^2a_j\cos jx_j\right)dx_j.$$

Searc in advance the value of A and B as next:

$$A=\frac{2}{\pi}\int_0^{\pi}\left(\sum_{k=1}^{K}k^2a_{kj}\cos kx_j\right)^2 dx_j$$

$$=\frac{2}{\pi}\sum_{k=1}^{K}k^4a_k^2\int_0^{\pi}\cos^2 kx_j dx_j$$

$$=\frac{1}{\pi}\sum_{k=1}^{K}k^4a_k^2\left[x_j+\frac{2}{k}\sin kx_j\right]_0^{\pi}$$

$$=\sum_{k=1}^{K}k^4a_{jk}^2 \qquad (6)$$

The next value of B is let by:

$$B = \frac{4}{\pi}\int_0^{\pi}\sum_{k<j}^{K}\left(k^2 a_{kj}\cos kx_j\right)\left(j^2 a_j \cos jx_j\right)dx_j$$
$$= \frac{4}{\pi}\sum_{k<j}^{K}\left(kj\right)^2 a_k a_j \int_0^{\pi}\left(\cos kx_j \cos jx_j\right)dx_j$$
$$= \frac{4}{\pi}\sum_{k<j}^{K}\left(kj\right)^2 a_k a_j \left[\frac{1}{k+j}\sin\left(k+j\right)x_j + \frac{1}{k-j}\sin\left(k-j\right)x_j\right]_0^{\pi}$$
$$= 0 \qquad (7)$$

Based on the equation (6) and (7) were obtained:

$$J\left(g_j\right) = A + B$$
$$= \sum_{k=1}^{K} k^4 a_{jk}^2 \qquad (8)$$

Lemma 1 can also be presented in the form of a matrix. Matrix form Lemma 1 is let the result of 1.

Result 1: *If* $J\left(g_j\right)$ *let of equation (6.8), then:*

$J\left(g_j\right) = \underset{\sim}{\beta}'\lambda^* D^* \underset{\sim}{\beta}$, when,

$$\lambda^* = \begin{pmatrix} \lambda_1 I_{k+2} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{k+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r I_{k+r} \end{pmatrix},\ D^* = \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{pmatrix} = diag\left(D, D, ..., D\right),$$

$$\beta = \begin{pmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \\ \vdots \\ \underset{\sim}{\beta}_r \end{pmatrix} \text{ and } D^* = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K^4 \end{pmatrix}.$$

Proof:
Based on Lemma 1 is obtained:

$$J\left(g_j\right) = \sum_{k=1}^{K} k^4 a_{kj}^2$$

With attention to this equation, Penalty part of the optimization of the equation (5) can be represented as:

$$\sum_{j=1}^{q}\lambda_j \int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2 dx_{ji} = \lambda_1\int_0^{\pi}\left[g_1''\left(x_1\right)\right]^2 + \lambda_2\int_0^{\pi}\left[g_2''\left(x_2\right)\right]^2 + ... + \lambda_q\int_0^{\pi}\left[g_q''\left(x_q\right)\right]^2$$
$$= \left(\lambda_1, \lambda_2, ..., \lambda_q\right)\begin{bmatrix} \sum_{k=1}^{K} k^4 a_{1k}^2 \\ \sum_{k=1}^{K} k^4 a_{2k}^2 \\ \vdots \\ \sum_{k=1}^{K} k^4 a_{qk}^2 \end{bmatrix}$$

$$= \lambda_1 \underset{\sim}{\beta}_1' D^* \underset{\sim}{\beta}_1 + \lambda_2 \underset{\sim}{\beta}_2' D^* \underset{\sim}{\beta}_2 + ... + \lambda_q \underset{\sim}{\beta}_q' D^* \underset{\sim}{\beta}_q$$

$$= \underset{\sim}{\beta}' \lambda^* D^* \underset{\sim}{\beta} \quad (9)$$

The addition to need Lemma 1, and result 1, theorem 1 also need Lemma 2 as next:

Lemma 2. *If* g_j *the approximated by Fourier series nontrend function (2), the goodness of fit* $R\left(g_j\right)$ *can be presented in the form of:*

$$R(g) = n^{-1}\left(\underset{\sim}{y} - X(K)\underset{\sim}{\beta}\right)'\left(\underset{\sim}{y} - X(K)\underset{\sim}{\beta}\right),$$

with,

$\underset{\sim}{y} = \left[y_1, y_2, ..., y_n\right]'$,

$\underset{\sim}{\beta} = \left[\frac{1}{2}a_{01}, a_{11}, ..., a_{K1} \vdots \cdots \vdots \frac{1}{2}a_{01q}, a_{1q}, ..., a_{Kq}\right]'$ *and*

$$X(K) = \begin{bmatrix} 1 & \cos x_{11} & \cdots & \cos Kx_{11} & \vdots & \cdots & \vdots & 1 & \cos x_{q1} & \cdots & \cos Kx_{q1} \\ 1 & \cos x_{12} & \cdots & \cos Kx_{12} & \vdots & \cdots & \vdots & 1 & \cos x_{q2} & \cdots & \cos Kx_{q2} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos x_{1n} & \cdots & \cos Kx_{1n} & \vdots & \cdots & \vdots & 1 & \cos x_{qn} & \cdots & \cos Kx_{qn} \end{bmatrix}$$

Proof:

$$R\left(\underset{\sim}{g}\right) = n^{-1}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{q} g_j\left(x_j\right)\right)^2$$

$$= n^{-1}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{q}\left(\frac{1}{2}a_{0j} + a_{1j}\cos x_{ji} + ... + a_{Kj}\cos Kx_{ji}\right)\right)^2$$

$$= n^{-1}\sum_{i=1}^{n}\left(y_i - \left(\frac{1}{2}a_{01} + a_{11}\cos x_{1i} + ... + a_{K1}\cos Kx_{1i}\right) - ... + \right.$$

$$\left. - \left(\frac{1}{2}a_{0q} + a_{1q}\cos x_{qi} + ... + a_{Kq}\cos Kx_{qi}\right)\right)^2 \quad (10)$$

With outline in the form of a matrix equation (10) can be written into:

$$R\left(\underset{\sim}{g}\right) = n^{-1}\left(\underset{\sim}{y} - X(K)\underset{\sim}{\beta}\right)'\left(\underset{\sim}{y} - X(K)\underset{\sim}{\beta}\right) \quad \blacksquare(11)$$

The using Lemma 1, result 1 and Lemma 2 forms Deret Fourier estimator nontrend let by Theorem 1 as next:

Teorema 2. *If let the regression model presented in the form of equation (1) and the Fourier series estimator nontrend obtained from optimization (5), then the estimator to function* $g_j(x_j), j = 1, 2, ..., q$ *let by:*

$$\hat{g}_j\left(x_j\right) = \frac{1}{2}\hat{a}_{oj} + \sum_{k=1}^{K}\hat{a}_{kj}\cos kx_{ji},$$

As a result, estimation for the regression curve $\mu\left(x_{1i}, x_{2i}, ..., x_{qi}\right)$ *let by:*

$$\hat{\mu}\left(x_{1i}, x_{2i}, ..., x_{qi}\right) = \sum_{j=1}^{q}\hat{g}_j(x_{ji}) = \sum_{j=1}^{q}\left(\frac{1}{2}\hat{a}_{oj} + \sum_{k=1}^{K}\hat{a}_{kj}\cos kx_{ji}\right).$$

Proof:
Nontrend Fourier series estimator obtained from optimization:

$$\underset{B\in R^{q(k+j)}}{Min}\left\{n^{-1}\sum_{i=1}^{n}\left(y_i-\sum_{j=1}^{q}g_j\left(x_{ji}\right)\right)^2+\sum_{j=1}^{q}\lambda_j\int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2dx_{ji}\right\}. \tag{12}$$

Based on Lemma 2 obtained:

$$R\left(\underset{\sim}{g}_j\right)=n^{-1}\sum_{i=1}^{n}\left(y_i-\sum_{j=1}^{q}g_{ji}\left(x_{ji}\right)\right)^2=n^{-1}\left(\underset{\sim}{y}-X\left(K\right)\underset{\sim}{\beta}\right)'\left(\underset{\sim}{y}-X\left(K\right)\underset{\sim}{\beta}\right)$$

The next by Lemma 1 and Result 1 obtained:

$$J\left(g_j\right)=\sum_{j=1}^{q}\lambda_j\int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2dx_{ji}=\underset{\sim}{\beta}'\lambda^*D^*\underset{\sim}{\beta}.$$

With combination the goodness of fit $R\left(g_j\right)$ and penalty $J\left(g_j\right)$, optimization (12) can be presented in the form of:

$$\begin{aligned}\underset{B\in R^{q(k+j)}}{Min} &= \underset{B\in R^{q(k+j)}}{Min}\left\{n^{-1}\left(\underset{\sim}{y}-X\left(K\right)\underset{\sim}{\beta}\right)'\left(\underset{\sim}{y}-X\left(K\right)\underset{\sim}{\beta}\right)+\underset{\sim}{\beta}'\lambda^*D^*\underset{\sim}{\beta}\right\}\\ &= \underset{B\in R^{q(k+j)}}{Min}\left\{Q\left(\underset{\sim}{\beta}\right)\right\}\end{aligned}$$

Estimator $\underset{\sim}{\beta}$ obtained by the partial deffrensiable $Q\left(\underset{\sim}{\beta}\right)$ to $\underset{\sim}{\beta}$. The first is described shape $Q\left(\underset{\sim}{\beta}\right)$ as next:

$$\begin{aligned}Q\left(\underset{\sim}{\beta}\right)&=n^{-1}\left(\underset{\sim}{y}-X\left(K\right)\underset{\sim}{\beta}\right)'\left(\underset{\sim}{y}-X\left(K\right)\underset{\sim}{\beta}\right)+\underset{\sim}{\beta}'\lambda^*D^*\underset{\sim}{\beta}\\ &=n^{-1}\left(\underset{\sim}{y}'\underset{\sim}{y}-\underset{\sim}{y}'X\left(K\right)\underset{\sim}{\beta}-\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{y}+\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{\beta}\right)+\underset{\sim}{\beta}'\lambda^*D^*\underset{\sim}{\beta}\\ &=n^{-1}\underset{\sim}{y}'\underset{\sim}{y}-n^{-1}\left(X'\left(K\right)\underset{\sim}{\beta}'\underset{\sim}{y}\right)'-n^{-1}\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{y}+n^{-1}\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{\beta}\\ &\quad+\underset{\sim}{\beta}'\left(n^{-1}X'\left(K\right)X\left(K\right)+\lambda^*D^*\right)\underset{\sim}{\beta}\\ &=n^{-1}\underset{\sim}{y}'\underset{\sim}{y}-2n^{-1}\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{y}+n^{-1}\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{\beta}+\underset{\sim}{\beta}'\left(n^{-1}X'\left(K\right)X\left(K\right)+\lambda^*D^*\right)\underset{\sim}{\beta}\end{aligned}$$

To minimize $Q\left(\underset{\sim}{\beta}\right)$ was obtained by minimizing the partial $Q\left(\underset{\sim}{\beta}\right)$ to $\underset{\sim}{\beta}$ and the result is equated with the zero equation:

$$\begin{aligned}\frac{\partial Q\left(\underset{\sim}{\beta}\right)}{\partial\underset{\sim}{\beta}}&=-2n^{-1}X'\left(K\right)\underset{\sim}{y}+2n^{-1}\underset{\sim}{\beta}'X'\left(K\right)\underset{\sim}{\beta}+2\left(n^{-1}X'\left(K\right)X\left(K\right)+\lambda^*D^*\right)\underset{\sim}{\beta}\\ &=2\left(-n^{-1}X'\left(K\right)\underset{\sim}{y}+\left(n^{-1}X'\left(K\right)X\left(K\right)+n^{-1}X'\left(K\right)X\left(K\right)+\lambda^*D^*\right)\underset{\sim}{\beta}\right)\end{aligned}$$

By equating the $\frac{\partial Q\left(\underset{\sim}{\beta}\right)}{\partial\underset{\sim}{\beta}}=0$ equation obtained as next:

$$\begin{aligned}\left(-n^{-1}X'\left(K\right)\underset{\sim}{y}+\left(n^{-1}X'\left(K\right)X\left(K\right)+n^{-1}X'\left(K\right)X\left(K\right)+\lambda^*D^*\right)\underset{\sim}{\beta}\right)&=0\\ \left(-X'\left(K\right)\underset{\sim}{y}+\left(X'\left(K\right)X\left(K\right)+X'\left(K\right)X\left(K\right)+n\lambda^*D^*\right)\underset{\sim}{\beta}\right)&=0\end{aligned}$$

With assume elaboration and matrix $X\left(K\right)$ nonsingular (matrik with full rank) then obtained estimator $\underset{\sim}{\beta}$ as next:

$$\hat{\underset{\sim}{\beta}}(K)=\left(X'(K)X(K)+X'(K)X(K)+n\lambda^{*}D^{*}\right)^{-1}X'(K)\underset{\sim}{y}$$

$$=\left[\frac{1}{2}\hat{a}_{01},\hat{a}_{11},...,\hat{a}_{K1}\vdots\cdots\vdots\frac{1}{2}\hat{a}_{01q},\hat{a}_{1q},...,\hat{a}_{Kq}\right]'$$

Estimator for the regression curve $g_j(x_j)$ let by:

$$\hat{g}_j\left(x_j\right)=\frac{1}{2}\hat{a}_{oj}+\sum_{k=1}^{K}\hat{a}_{kj}\cos kx_{ji},$$

A result estimates for the regression curve $\mu\left(x_{1i},x_{2i},...,x_{qi}\right)$ let by:

$$\hat{\mu}\left(x_{1i},x_{2i},...,x_{qi}\right)=\sum_{j=1}^{q}\hat{g}_j\left(x_j\right)$$

$$=\sum_{j=1}^{q}\left(\frac{1}{2}\hat{a}_{oj}+\sum_{k=1}^{K}\hat{a}_{kj}\cos kx_{ji}\right)$$ ■

CONCLUSIONS

Based on the result and discussion that has been done so can be concleded as next:
Nonparametric regression curve estimator multivariable Fourier series nontrend additives derived from optimization:

$$\underset{B\in R^{q(k+j)}}{Min}\left\{n^{-1}\sum_{i=1}^{n}\left(y_i-\sum_{j=1}^{q}g_j\left(x_{ji}\right)\right)^2+\sum_{j=1}^{q}\lambda_j\int_0^{\pi}\frac{2}{\pi}g_j''\left(x_{ji}\right)^2dx_{ji}\right\}.$$

This optimization result in Fourier series estimator nontrend:

$$\hat{\mu}\left(x_{1i},x_{2i},...,x_{qi}\right)=\sum_{j=1}^{q}\hat{g}_j\left(x_j\right)=\sum_{j=1}^{q}\left(\frac{1}{2}\hat{a}_{oj}+\sum_{k=1}^{K}\hat{a}_{kj}\cos kx_{ji}\right)$$

with $\hat{a}_{oj}$, $\hat{a}_{kj}$, j=1,2,...,q, k=1,2,...,K let by equations:

$$\hat{\underset{\sim}{\beta}}(K)=\left(X'(K)X(K)+X'(K)X(K)+n\lambda^{*}D^{*}\right)^{-1}X'(K)\underset{\sim}{y}$$

$$=\left[\frac{1}{2}\hat{a}_{01},\hat{a}_{11},...,\hat{a}_{K1}\vdots\cdots\vdots\frac{1}{2}\hat{a}_{01q},\hat{a}_{1q},...,\hat{a}_{Kq}\right]'.$$

REFERENCES

1. Bilodeau, M., Fourier Smoother and Additive Models, *The Canadian Journal of Statistics* **3**, 257-269 (1992).
2. Budiantara, I. N., Lestari, B., Islamiyati, A. , *Estimator Spline Terbobot dalam Regresi Nonparametrik dan Semiparametrik Heteroskedatisitas untuk Data Longitudinal*. DP2M Dikti. Jakarta, 2010.
3. De Jong, P., The Fast Fourier Transform Spectral Estimator, *Journal of Statistical SOC.B 39,* **3**, 327-330 (1977).
4. Eubank, R. L., *Nonparametric Regression and Spline Smoothing*, Second Edition, New York, Marcal Dekker, Inc, 1999.
5. Faber, D., Douglas, C. Yoon, Susan K., Service and Struat C. White., Fourier and Wavelet Analysis of Dental Radiographs Detect Trabecular Changes in Osteoporosis Bone, 35, 403-411, 2004.
6. Galtchoulk, L. and Pargemenshcikov, S., Adaptive Asymptotically Efficient Estimator in Heterosceastic Nonparametric Regression, *Journal of The Korea Statistical Society* **38**, 305-322 (2009).
7. He, H. and Huang, L. S. , Double Smoothing for Bias Reduction in Lokal Linier Regression, *Journal of Statistical Planning and Inference* **139**, 1056-1072 (2009).

8. Kayri, M., and Zirhloglu, G., Kernel Smoothing and Coosing Bandwith for Nonparametric Regression Methods, *Ozean Journal of Applied Sciences*, **2**, 49-54 (2009).
9. Liang, H., Estimation in Partially Linier Models and Numerical Comparisons, *Computational Statistics and Data Analysis*, **50**, 675-687, DOI 10.101617.CSDA.2004.10.007 (2006).
10. Mahler, Variational Solution of Penalized Likelihood Problem and Smooth Curva, *Annal of Statistic* **23**, 1496-1517 (1995).
11. Manzan, S. and Zerom, D., Kernel Estimation of a Partially Linier Additive Model, *Statistic and Probability Letters*, **72**, 313-322 (2005).
12. Morton, J., Silverberg, L., 2009, Fourier Series of Half-Range Functions by Smooth Extension, Applied Mathematical Modelling, 33(2009) 812-821.
13. Qingguo, T., 2010. Li-Estimation in a Semiparametric Model with Longitudinal Data, Journal of Statistic Planning and Inference, 140, 393-405.
14. Okumura, H. And Naito, K., Nonparametric Kernel Regression for Multinomial Data, *Journal of Multivariate Analysis*, **97**, 2009-2022 (2006).
15. Qu, L., On Semiparametric Regression Via Wavelet, Disertation, Purduc University, 2003.
16. Rakotomamonjy, K., Mary, X. and Conu, S., Nonparametric Regression with Wavelet Kernels, *Appl.. Statistic Models in Business. and Industry*., **21**, 153-163, 10.1002/asmb.533 (2005).
17. Su, L., and Ullah, A.. Local Polynomial Estimation of Nonparametric Simutancous Equations Models, *Journal of Econometrics*, **144**, 193-218 (2008).
18. Taylor, L., W. 2009, Using the Haar Wavelet Transform in the Semiparametric Speciofication of Time Series, Economic Modelling, 26, 392-403.
19. Tripena, A. And Budiantara, I N., Fourier Estimator in Nonparametric Regression, *International Conference on Natural Sciences and Applied Natural Scienes*, Ahmad Dahlan University, Yogyakarta, 2007.
20. Yao, F., 2007, Asymptotic Distribution of Nonparametric Regresion Estimators for Longitudinal or Functional Data, Journal of Multivariate Analysis, 98, 40-56.

Proceedings of "The 7^{th} SEAMS-UGM Conference 2015"
Topic, pp. 84–86.

Invariance of p-Regular Subspace of Some Symmetric Nakayama Algebras

Intan Muchtadi-Alamsyah* and Aditya Purwa Santika†

*Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung,
Email: ntan@math.itb.ac.id
†Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung

Abstract. This article deals with a substructure of the center of symmetric Nakayama algebra called p-regular subspace. We prove that this subspace is invariant under derived self-equivalence using explicit two-sided tilting complexes.

Keywords: symmetric Nakayama algebra, p-regular subspace, derived equivalence, two-sided tilting complex

INTRODUCTION

For a field F of characteristic prime $p > 0$ and a finite group G, the p-regular subspace $Z_{p'}FG$ of the group algebra FG is the F-subspace spanned by sums of p-regular conjugation classes, i.e. conjugation classes of elements whose order is not divisible by p. The dimension of this subspace is related to the number of isomorphism classes of simple modules over the group algebra FG. At first, its definition solely depends on the order of elements in group.

However, Santika and Zimmermann in [5] identify p-regular subspace as the dual of intersection of the images of p^n-power maps. This identification is adapted as definition of p-regular subspace in any symmetric algebras. Santika and Muchtadi-Alamsyah provided examples of p-regular subspaces of symmetric Nakayama algebras, algebras of dihedral type and algebras of semi-dihedral type in [4].

Broué's conjecture has motivated some research on group representation theory. Some of the research work is to prove some invariance of a structure under derived equivalence. One of them is the invariance of Külshammer ideal under derived equivalence, shown by Zimmermann in[6].

By using the identification of p-regular subspace as the dual of intersection of the images of p^n-power maps, invariance of p-regular subspace under derived equivalence has been studied in [5] and for symmetric algebra in [3]. As derived equivalence can be given by standard two-sided tilting complexes (see [2]), in this paper we prove that the p-regular subspace of some symmetric Nakayama algebra is invariant under derived self-equivalence by using explicit two-sided tilting complexes.

THE p-REGULAR SUBSPACES

Let p be a prime number, G a finite group and F a field with characteristic $p > 0$. An element g in G is called a p-element if its order is p^n for some integer n and p'-element if its order is not divisible by p. Let $C_1, C_2, \cdots, C_n$ be the conjugation classes of G. The conjugation class C_i is a p-regular class if C_i consists of p'-elements. Define the class sum of C_i as the sum of all elements in C_i, i.e.

$$C_i^+ = \sum_{c \in C_i} c, \quad \text{for all } i \in \{1, \cdots, n\}.$$

It can be shown that $\{C_1^+, \cdots, C_n^+\}$ is a basis of $Z(FG)$, the center of the group algebra FG.

Let $C_1, \cdots, C_k$ ($k \leq n$) be the p-regular classes. The subspace generated by $\{C_1^+, \cdots, C_k^+\}$ is called the p-regular classes, denoted by $Z_{p'}(FG)$.

For any algebra A we define $[A,A]$ as the commutator subspace of A If A is a symmetric algebra with symmetrizing form $\langle\ ,\ \rangle$, then for any basis of $Z(A)$ (the center of A) we get a basis of $A/[A,A]$. Therefore we get an identification $\delta : Z(A) \to A/[A,A]$.

Define the p-power map $\mu_p : FG/[FG,FG] \to FG/[FG,FG]$ with $\mu_p(a) = a^p$ for every $a \in FG/[FG,FG]$. By [5, Lemma 2] an element C_h^+ is in the basis of $Z_{p'}(FG)$ if and only if

$$h + [FG,FG] \in \cap_{t\in\mathbb{N}} im(\mu_p^t).$$

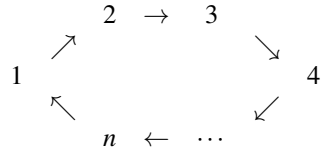
As this condition does not depend on the order of elements of G, we can apply this condition to define the p-regular subspace of any symmetric algebra.

Definition 1 *Let A be a symmetric algebra and let $\mu_p : A/[A,A] \to A/[A,A]$ with $\mu_p(a) = a^p$ for every $a \in A/[A,A]$. The p-regular subspace $Z_{p'}(A)$ of $Z(A)$ is the pre-image of $\cap_{t\in\mathbb{N}} im(\mu_p^t)$ via the identification δ.*

INVARIANCE OF SOME SYMMETRIC NAKAYAMA ALGEBRAS

This article focuses on the symmetric Nakayama algebra N_n^n, which definition is given below.

Definition 2 *The symmetric Nakayama algebra N_n^n is the path algebra with quiver*



modulo ideal generated by all paths of length $n+1$.

In [4] we proved that the p-regular subspace of $A = N_n^n$ is

$$Z_{p'}(N_n^n) = span\{\alpha_i\alpha_{i+1}\cdots\alpha_{i-2}\alpha_{i-1}, 1 \le i \le n\}$$

and its dual in commutator quotient space $A/[A,A]$ is $span\{e_1, e_2, \ldots, e_n\}$.

Let us denote $A = N_n^n$ and for $i \in \{1,2,\cdots,n\}$ let P_i denote the indecomposable projective A-modules and ${}_iP = Hom_A(P_i,A)$. By [1] the following complexes are two-sided tilting complexes that give auto-equivalence of derived category of A :

$$F_i := 0 \longrightarrow P_i \otimes {}_iP \longrightarrow A \longrightarrow 0 \text{ and}$$

$$F_i' := 0 \longrightarrow A \longrightarrow P_i \otimes {}_iP \longrightarrow 0,$$

where A is in degree 0, for all $i \in \{1,2,\cdots,n\}$.

The following is the result of this paper.

Theorem 3 *Let $A := N_n^n$ be the symmetric Nakayama algebra over F and $Z_{p'}(A)$ be its p-regular subspace. If $\mathscr{F}_i = F_i \otimes_A - \otimes_A F_i'$ then*

$$\mathscr{F}_i(A/[A,A]) = (A/[A,A]) \text{ and } \mathscr{F}_i(Z_{p'}(A)) = Z_{p'}(A)$$

for all $i \in \{1,2,\cdots,n\}$.

Proof. Let $S_i = span\{e_i\}$ for all $i \in \{1,\cdots,n\}$. By computing $F_i \otimes_A S_i \otimes_A F_i'$ for all $i \in \{1,\cdots,n\}$ we get

$$\begin{aligned}
F_i \otimes_A S_i \otimes_A F_i' &= (0 \to P_i \otimes {}_iP \otimes_A S_i \to A \otimes_A S_i \to 0) \otimes_A (0 \to A \to P_i \otimes {}_iP \to 0) \\
&= (0 \to P_i \to S_i \to 0) \otimes_A (0 \to A \to P_i \otimes {}_iP \to 0) \text{ since } dim({}_iP \otimes_A S_i) = dim(Hom_A(P_i,S_i)) = 1 \\
&= 0 \to P_i \to (S_i \otimes_A A) \oplus (P_i \otimes {}_iP \otimes_A P_i) \to P_i \otimes {}_iP \otimes_A S_i \to 0 \text{ where the middle term is in degree 0} \\
&= 0 \to P_i \to S_i \oplus P_i \oplus P_i \to P_i \to 0 \text{ since } dim({}_iP \otimes_A P_i) = dim(Hom_A(P_i,P_i)) = 2 \\
&\cong S_i \text{ (homotopy equivalent).}
\end{aligned}$$

For $i \neq j$,

$$\begin{aligned} F_i \otimes_A S_j \otimes_A F_i' &= (0 \to P_i \otimes {}_iP \otimes_A S_j \to A \otimes_A S_j \to 0) \otimes_A (0 \to A \to P_i \otimes {}_iP \to 0) \\ &= (0 \to S_j \to 0) \otimes_A (0 \to A \to P_i \otimes {}_iP \to 0) \text{ since } {}_iP \otimes_A S_j = Hom_A(P_i, S_j) = 0 \\ &= 0 \to S_j \otimes_A A \to P_i \otimes {}_iP \otimes_A S_j \to 0 \\ &= S_j \end{aligned}$$

Now $A/[A,A] = span\{e_1, \cdots, e_n\} = \oplus_{i=1}^n S_i$. Therefore for all $i \in \{1, \cdots, n\}$,

$$\begin{aligned} \mathscr{F}_i(A/[A,A]) &= F_i \otimes_A A/[A,A] \otimes_A F_i' \\ &= F_i \otimes_A (\oplus_{k=1}^n S_k) \otimes_A F_i' \\ &= A/[A,A]. \end{aligned}$$

By [3, Theorem 4] the bilinear form will differ by some central unit in $Z(A)$, but since in symmetric Nakayama algebra A the only central unit is 1, therefore the bilinear form is preserved and $\mathscr{F}(Z_{p'}A) = Z_{p'}A$. The proof is complete.

CONCLUSION AND FURTHER RESEARCH

In this paper we prove that the p-regular subspace of some symmetric Nakayama algebras is invariant under derived self-equivalence given by some explicit two-sided tilting complexes. For further research we will investigate the invariance given by explicit two-sided tilting complexes which are not derived self-equivalence.

ACKNOWLEDGMENTS

This research is supported by Hibah Desentralisasi DIKTI 2015.

REFERENCES

1. I. Muchtadi-Alamsyah, *Comm. in Algebra* **36**, 2544-2569 (2008).
2. J. Rickard, *J. Pure Appl. Algebra* **61**, 303-317 (1989).
3. A. P. Santika, *Subruang p-Reguler dan Kajian Invariansinya terhadap Ekivalensi Bentukan*, PhD Dissertation, 2015.
4. A. P. Santika and I. Muchtadi-Alamsyah, *JP J. of Algebra, Number Theory and App.* **27**, 131-142 (2012).
5. A. P. Santika and A. Zimmermann, Derived equivalences and substructures of the centre related to p-power maps, preprint.
6. A. Zimmermann, *Math. Proc. of the Royal Irish Academy* **107A**, 1-9 (2007)

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 87–94.

The Properties of Intra-Regular and Left (Right) Weakly-Regular Ordered Bilinear Semigroups Related to Their Fuzzy Ideals

Karyati[1] and Agus Maman Abadi[2]

[1] *Mathematics Education Department, Yogyakarta State University,*
Email: karyati@uny.ac.id, jengkaruny@gmail.com
[2] *Mathematics Education Department, Yogyakarta State University,*
Email: agusmaman@uny.ac.id

Abstract. We obtain some properties of the intra-regular and left (right) weakly-regular ordered bilinear form semigroup, i.e.: if $(S(B)^1,\circ,\leq)$ has an identity element, then $S(B)^1$ is both intra-regular and left(right) weakly-regular if and only if for every fuzzy left ideal I_L^F, and for every fuzzy right ideal I_R^F and every fuzzy quasi-ideal Q^F of $S(B)^1$, we have a property related to these ideals: $I_L^F \wedge I_R^F \wedge Q^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc Q^F$. If $(S(B)^1,\circ,\leq)$ is both intra-regular and left (right) weakly-regular if and only if for every fuzzy left ideal I_L^F, and for every fuzzy right ideal I_R^F and every fuzzy quasi-ideal Q^F of $S(B)^1$, we have a property related to these ideals: $I_L^F \wedge I_R^F \wedge Q^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc Q^F$.

Keywords: intra-regular, bilinear semigroups, fuzzy ideals

INTRODUCTION

Theory of fuzzy subset has been established by Zadeh. The developing of this theory has been done by many researchers. Rosenfeld has been developed this theory to the fuzzy subgroupoid theory. Zimmerman has consider the application of the fuzzy subsets. In [11], Mordeson and Malik has developed the fuzzy subset theory on fuzzy semigroup. Karyati et al. [6] have been developed the theory of subsemigroup fuzzy into the special semigroup called fuzzy bilinear form subsemigroup. The new theory has been establish, i.e.: the characteristics of fuzzy right/left ideal, the fuzzy principle ideal, fuzzy relation and Green relation on bilinear form semigroups. Kehayopulu et al. [9] have established the theory of the partial ordered semigroup and groupoid. In [8], Karyati and Dhoriva have established the characteristics of the partial ordered bilinear form semigroup in term of their fuzzy subsets.

A semigroup $(S,.)$ with a *partial order* operation $'\leq'$, such that $(S,\leq)$ is a partial ordered set (poset) and for every $x,y,z \in S$, with $x \leq y$, we have $zx \leq zy$ and $xz \leq yz$, then $(S,.,\leq)$ is called partial ordered semigroup. Many researchers have reseach about this topic. Defining a partial order into a semigroup has many consequences. These are related to the defining of (right/left) ideal, right/left) quasi-ideal, fuzzy (right/left) ideal and fuzzy (right/left) quasi-ideal. Based on these definitions, we can develope to get the new theories related to the partial ordered semigroups. In this paper, we will find the characteristics of the partial bilinear form semigroup in term their right and left ideals, quasi ideals and fuzzy bi-ideals.

THEORYTICAL REVIEW

On this section, we give many definitions, theorems, lemmas, propositions and corollaries to support this research. All of these theories related to the ordered semigroup. Based on the definition of an ordered semigroup, we will give the new definition about the (left / right) ideal of an ordered semigroup, quasi ideal semigroup and

bi-ideal semigroup. Further, we serve the definition of fuzzy (left/right) ideal, fuzzy quasi ideal, fuzzy bi-ideal of an ordered semigroup.

Partial Ordered Semigroup (*po_semigrup*)

A semigroup is an algebra structure with an associative binary operation. A non empty set S with a binary operation $'.'$ is called a semigroup if $(\forall x, y \in S)\ x.y \in S$ and $(\forall x, y, z \in S)\ (x.y).z = x.(y.z)$. An element of a semigroup S a is called a regular element if there exist $a' \in S$ such that $a = aa'a$. A semigroup S is called a regular semigroup if and only if every element of S is a regular element. A non empty set P is called partial ordered $' \leq '$ if and only if: reflective, i.e $(\forall x \in P) x \leq x$, anti symmetry *i.e.* $(\forall x, y \in P)\ x \leq y$ and $y \leq x \Longrightarrow x = y$, transitive i.e. $(\forall x, y, z \in P)\ x \leq y$ and $y \leq z \Longrightarrow y \leq z$. The partial partial ordered set is called a poset. The following definition give a definition about a partial ordered semigroup:

Definition 1. Let S *be a non empty set. The set* S *with a binary operation* $'.'$ *and a partial ordered* $' \leq '$ *is called a partial ordered semigroup if and only if:*

i. $(S, .)$ *is a semigroup*

ii. $(S, \leq)$ *is a partial ordered set*

iii. $(\forall a, b, x \in S) a \leq b \Longrightarrow xa \leq xb$ *and* $ax \leq bx$

Definition 2. *Let* $(S, ., \leq)$ *be a partial ordered semigroup. Then a non empty subset* I *is called an ideal of a semigroup* S *if :*

i. $(\forall a \in S)(\forall b \in I)\ a \leq b \Longrightarrow a \in I$

ii. $IS \subseteq I$ *and* $SI \subseteq I$

If $A \subseteq S$, for S is an ordered semigroup, then we denote $(A] = \{t \in S \mid t \leq h,$ for some $h \in A\}$. A nonempty subset A is a left (right) ideal if: (1) $SA \subseteq A$ $(AS \subseteq A)$ (2) If $a \in A$, $b \in S$ and $b \leq a$ implies $b \in A$. Two sided ideal (or it is just called ideal) is both a left and a right ideal. An ordered semigroup is called regular if for every $a \in S$ there exist $x \in S$ such that $a \leq axa$. It is equivalent with (1) $A \subseteq (ASA]$, $\forall A \subseteq S$, (2) $a \in (aSa]$, for every $a \in S$. An ordered semigroup is called intra-regular if for every $a \in S$ there exist $x, y \in S$such that $a \leq xa^2y$. This definition is equivalent with (1) $A \subseteq (SA^2S]$, for every $A \subseteq S$, (2) $a \in (SA^2S]$, for every $a \in S$. An ordered semigroup S is called left (right) weakly-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xaya$ $(a \leq axay)$. For every A, B of an ordered semigroup S, we have $A \subseteq (A]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$, $\big((A]\big] = (A]$. For $a \in S$, we denote $A_a = \{(y, z) \in S \times S |\ a \leq yz\}$.

Bilinear Form Semigroups

A bilinear form semigroup is a special semigroup. We give the following theory how to construct a bilinear form semigroup. Let $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ be a set of all linear operator X and Y, respectively. If $f \in \mathcal{L}(X)$, then we get a vector subspace of X:

$$N(f) = \{u \in X \mid f(u) = 0\} \text{ and } R(f) = \{v \in X \,\big|\, f(x) = v, \text{ for any } x \in X\}$$

An element $f \in \mathcal{L}(X)$ is called an adjoin pair with $g \in \mathcal{L}(Y)$with respect to the bilinear form B, and vice versa, if and only if $B(x, g(y)) = B(f(x), y)$ for every $x \in X$ and $y \in Y$. The next, we will denote the following sets:

$$\mathcal{L}'(X) = \big\{f \in \mathcal{L}(X) \big| N(B_*) \subseteq N(f),\ \ R(f) \cap N(B_*) = \{0\}\big\}$$
$$\mathcal{L}'(Y) = \big\{g \in \mathcal{L}(Y) \big| N(B^*) \subseteq N(g),\ \ R(g) \cap N(B^*) = \{0\}\big\}$$
$$S(B) = \{(f, g) \in \mathcal{L}'(X) \times \mathcal{L}'(Y)^{op} | (f, g) \text{ an adjoin pair}\}$$

Karyati et al. (2002) have proved that the set $S(B)$ is a semigroup with respect to the binary operation which is defined as $(f, g)(f', g') = (ff', g'g)$, [4]. This semigroup $S(B)$ is called a bilinear form semigroup.

The properties of this semigroup has been establish by Rajendran & Nambboripad in [18]. Based on this properties, Karyati et al. in [5], [6], [7], [8], [9], [10], [11] have developed this theory included the fuzzy version.

Fuzzy Subsemigroups

Refer to the papers which are written by Asaad in [1], Kandasamy in [3], Mordeson and Malik in [16], Shabir in [19], we have a definition of a fuzzy subset α of a semigroup S is a mapping from S into $[0,1]$,i.e. $\alpha: S \to [0,1]$.

Definition 3. *Let S be a semigroup. A mapping* $\alpha: S \to [0,1]$ *is called a fuzzy subsemigroup if and only if* $\alpha(xy) \geq \min \{\alpha(x), \alpha(y)\}$ *for every* $x, y \in S$.

Definition 4. [15] *Let* α *be a fuzzy subsemigrup of a semigroup S. Then:*
(i) α *is a fuzzy left ideal if* $(\forall x, y \in S)\ \alpha(xy) \geq \alpha(y)$
(ii) α *is a fuzzy rigth idea lif* $(\forall x, y \in S)\alpha(xy) \geq \alpha(x)$
(iii) α *is a fuzzy ideal if* α *is a fuzzy left ideal and a fuzzy right ideal, i.e.:* $(\forall x, y \in S)\ \alpha(xy) \geq max\ \{\alpha(x), \alpha(y)\}$

Let S be a partial ordered semigroup. Then the definition of a fuzzy left ideal, fuzzy right ideal and fuzzy ideal (two sided) of S are defined as follow:

Definition 5. [15] *Let* $(S, ., \leq)$ *be a partial ordered semigroup . Then a fuzzy suset* α *of the partial ordered semigroup* S *is called fuzzy left ideal if :*

i. $(\forall x, y \in S)\ \alpha(xy) \geq \alpha(y)$
ii. $(\forall x, y \in S)\ x \leq y \Longrightarrow \alpha(x) \geq \alpha(y)$

Definition 6. [15] *Let* $(S, ., \leq)$ *be a partial ordered semigroup . Then a fuzzy subset* α *of the partial ordered semigroup* S *is called fuzzy right ideal if :*
i. $(\forall x, y \in S)\ \alpha(xy) \geq \alpha(x)$
ii. $(\forall x, y \in S)\ x \leq y \Longrightarrow \alpha(x) \geq \alpha(y)$

RESULTS

Lemma 1. *Let* $(S(B)^1, \circ, \leq)$ *be an ordered bilinear form semigroup with identity element '1'. Then we have equivalent statements:*

i. $S(B)^1$ *is both intra-regular and left weakly regular*
ii. $I_L \cap I_R \cap Q \subseteq (I_L I_R Q]$ *for every quasi-ideal* Q *every left ideal* I_L *and every right ideal* I_R *of the semigroup* $S(B)^1$
iii. $I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x}) I_R(\tilde{x}) Q(\tilde{x})]$ *for every* $\tilde{x} \in S(B)^1$.

Proof.

$(i \Longrightarrow ii)$ In this case we have $S(B)^1$ is both intra-reguler and left weakly regular. Then we will prove that $I_L \cap I_R \cap Q \subseteq (I_L I_R Q]$ for every quasi-ideal Q every left ideal I_L and every right ideal I_R of the semigroup $S(B)^1$. First,for every $\tilde{x} \in I_L \cap I_R \cap Q$, we have $\tilde{x} \in I_L$, $\tilde{x} \in I_R$ and $\tilde{x} \in Q$. We must remember that $S(B)^1$ is intra-regular, then there exist $\tilde{y}, \tilde{z} \in S(B)^1$ such that $\tilde{x} \leq \tilde{y}\tilde{x}^2\tilde{z}$. On the other hand $S(B)^1$ is left weakly regular, so there exist $\tilde{u}, \tilde{v} \in S(B)^1$ such that $\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x}$. Hence, we get:

$$\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x} \leq \tilde{u}(\tilde{y}\tilde{x}\tilde{x}\tilde{z})\tilde{v}\tilde{x} = \big((\tilde{u}\tilde{y})\tilde{x}\big)\big((\tilde{x}\tilde{z}\tilde{v})\tilde{x}\big) \in (S(B)^1 I_L)(I_R S(B)^1) \subseteq I_L I_R Q \subseteq (I_L I_R Q].$$

Finally we prove that $I_L \cap I_R \cap Q \subseteq (I_L I_R Q]$

$(ii \Longrightarrow iii)$ The second, we must prove $I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x}) I_R(\tilde{x}) Q(\tilde{x})]$ for every $\tilde{x} \in S(B)^1$ if it is known $I_L \cap I_R \cap Q \subseteq (I_L I_R Q]$ for every quasi-ideal Q every left ideal I_L and every right ideal I_R of the semigroup $S(B)^1$. Take an arbitrary element $\tilde{x} \in S(B)^1$, then we obtain $I_L(\tilde{x})$ is a left ideal generated by $\tilde{x}$, $I_R(\tilde{x})$ is a right ideal generated by $\tilde{x}$ and $Q(\tilde{x})$ is a quasi-ideal generated by $\tilde{x}$. On the other hand, we have $I_L \cap I_R \cap Q \subseteq (I_L I_R Q]$.

Finally we have

$I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]$.

$(iii \Longrightarrow i)$ The third, we have $I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]$ for every $\tilde{x} \in S(B)^1$. Then we will prove $S(B)^1$ is both intra-regular and left weakly regular. Take an arbitrary element $\tilde{x} \in S(B)^1$, then we have:

$$\tilde{x} \in I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})] \subseteq (I_L(\tilde{x})I_R(\tilde{x})S] \subseteq (I_L(\tilde{x})I_R(\tilde{x})]$$
$$= \big((S(B)^1\tilde{x}](\tilde{x}\, S(B)^1]]$$
$$= \Big(\big((S(B)^1\tilde{x})(\tilde{x}\, S(B)^1)\big]\Big] = \big((S(B)^1\tilde{x})(\tilde{x}\, S(B)^1)\big] = (S(B)^1\tilde{x}^2\, S(B)^1].$$

Hence, $S(B)^1$ is both intra-regular.

The other case, we have:

$\tilde{x} \in I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})] = \big((S(B)^1\tilde{x}](\tilde{x}\, S(B)^1](S(B)^1\tilde{x} \cap \tilde{x}\, S(B)^1]\big] \subseteq$

$\big((S(B)^1\tilde{x})(\tilde{x}\, S(B)^1)(S(B)^1\tilde{x} \cap \tilde{x}\, S(B)^1]\big] = \big((S(B)^1\tilde{x})(\tilde{x}\, S(B)^1)(S(B)^1\tilde{x} \cap \tilde{x}\, S(B)^1]\big] =$

$\big((S(B)^1\tilde{x}^2\, S(B)^1)(S(B)^1\tilde{x} \cap \tilde{x}\, S(B)^1]\big] = (S(B)^1\tilde{x}^2(S(B)^1)^2 \cap S(B)^1\tilde{x}^2\, S(B)^1\tilde{x}\, S(B)^1] \subseteq$

$(S(B)^1\tilde{x}S(B)^1\tilde{x} \cap S(B)^1\tilde{x}S(B)^1\tilde{x}S(B)^1] \subseteq (S(B)^1\tilde{x}S(B)^1\tilde{x}]$.

So, we obtain $S(B)^1$ left weakly regular. ■

Lemma 2. *Let $(S(B)^1,\circ,\leq)$ be an ordered bilinear form semigroup with identity element '1'. The semigroup $(S(B)^1,\circ,\leq)$ is both intra regular and left weakly regular if and only if for every fuzzy left ideal I_L^F, fuzzy right ideal I_R^F and fuzzy quasi-ideal Q^F of $S(B)^1$, we obtain:*

$$I_L^F \wedge I_R^F \wedge Q^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc Q^F$$

Proof.

If we have I_L^F is a fuzzy left ideal, I_R^F is fuzzy right ideal and Q^F is a fuzzy quasi-ideal of $S(B)^1$, respectively, then we obtain:

$$(I_L^F \wedge I_R^F \wedge Q^F)(\tilde{x}) \leq (I_L^F \circledcirc I_R^F \circledcirc Q^F)(\tilde{x}), \text{ for all } \tilde{x} \in S(B)^1.$$

On the other hand we have $S(B)^1$ is intra-regular. Based on the definition of intra-regular, so there exist $\tilde{y}, \tilde{z} \in S(B)^1$ such that $\tilde{x} \leq \tilde{y}\tilde{x}^2\tilde{z}$. We have $S(B)^1$ is left weakly regular too. Then there exist $\tilde{u}, \tilde{v} \in S(B)^1$ such that $\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x}$. Based on these two conditions, then we have:

$$\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x} \leq \tilde{u}(\tilde{y}\tilde{x}^2\tilde{z})\tilde{v}\tilde{x} = ((\tilde{u}\tilde{y})\tilde{x})(\tilde{x}(\tilde{z}\tilde{v})\tilde{x}).$$

We have $((\tilde{u}\tilde{y})\tilde{x})(\tilde{x}(\tilde{z}\tilde{v})\tilde{x}) \in A_{\tilde{x}}$. This proves that $A_{\tilde{x}} \neq \emptyset$. Thus:

$$(I_L^F \circledcirc I_R^F \circledcirc Q^F)(\tilde{x}) = \underset{(\tilde{p},\tilde{q})\in A_{\tilde{x}}}{\text{maks}}\{\min\{I_L^F(\tilde{p}), (I_R^F \circledcirc Q^F)(\tilde{q})\}\}$$
$$\geq \min\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), (I_R^F \circledcirc Q^F)(\tilde{x}(\tilde{z}\tilde{v})\tilde{x})\}$$
$$= \min\Big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), \underset{(\tilde{r},\tilde{s})\in A_{(\tilde{x}(\tilde{z}\tilde{v})\tilde{x})}}{\text{maks}}\{\min\{I_R^F(\tilde{r}), Q^F(\tilde{s})\}\}\Big\}$$
$$\geq \min\big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), \min\big\{I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big), Q^F(\tilde{x})\big\}\big\}$$
$$= \min\big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big), Q^F(\tilde{x})\big\}.$$

We know that I_L^F is a fuzzy left ideal of $S(B)^1$, so we have $I_L^F((\tilde{u}\tilde{y})\tilde{x}) \geq I_L^F(\tilde{x})$. And we know that I_R^F is a fuzzy right ideal of $S(B)^1$, so we have $I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big) \geq I_R^F(\tilde{x})$. So, we have:

$\min\big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big), Q^F(\tilde{x})\big\} \geq \min\{I_L^F(\tilde{x}), I_R^F(\tilde{x}), Q^F(\tilde{x})\} = (I_L^F \wedge I_R^F \wedge Q^F)(\tilde{x})$

Conversely, we have $I_L^F \wedge I_R^F \wedge Q^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc Q^F$, for every fuzzy left ideal I_L^F, fuzzy right ideal I_R^F and fuzzy quasi-ideal Q^F of $S(B)^1$. Then we must prove that $S(B)^1$ is both intra-regular and left weakly-regular.

Based on the previous Lemma 3.1, It is enough to prove that:

$$I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})], \text{ for every } \tilde{x} \in S(B)^1.$$

First, we take an arbitrary element $\tilde{x} \in S(B)^1$ and $\tilde{y} \in I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x})$. So, we have $\tilde{y} \in (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]$. We know that if I_L^F, I_R^F and Q^F are fuzzy left ideal, fuzzy right ideal and fuzzy quasi ideal, respectively, then $I_L(\tilde{x}), I_R(\tilde{x}), Q(\tilde{x})$ are left ideal, right ideal and quasi ideal generated by $\tilde{x}$, respectively. So the characteristic functions $C_{I_L(\tilde{x})}$, $C_{I_R(\tilde{x})}$, $C_{Q(\tilde{x})}$ are fuzzy left ideal, fuzzy right ideal and fuzzy quasi ideal of $S(B)^1$, respectively. Based on the hypothesis, we have:

$\left(C_{I_L(\tilde{x})} \wedge C_{I_R(\tilde{x})} \wedge C_{Q(\tilde{x})}\right)(y) \leq \left(C_{I_L(\tilde{x})} \circledcirc C_{I_R(\tilde{x})} \circledcirc C_{Q(\tilde{x})}\right)(\tilde{y})$.

We know that $\left(C_{I_L(\tilde{x})} \wedge C_{I_R(\tilde{x})} \wedge C_{Q(\tilde{x})}\right)(\tilde{y}) = \min\{C_{I_L(\tilde{x})}(\tilde{y}), C_{I_R(\tilde{x})}(\tilde{y}), C_{Q(\tilde{x})}(\tilde{y})\}$. So, we get :

$$\min\{C_{I_L(\tilde{x})}(\tilde{y}), C_{I_R(\tilde{x})}(\tilde{y}), C_{Q(\tilde{x})}(\tilde{y})\} \leq \left(C_{I_L(\tilde{x})} \circledcirc C_{I_R(\tilde{x})} \circledcirc C_{Q(\tilde{x})}\right)(\tilde{y}).$$

Since $\tilde{y} \in I_L(\tilde{x})$, $\tilde{y} \in I_R(\tilde{x})$, and $\tilde{y} \in Q(\tilde{x})$, so $C_{I_L(\tilde{x})}(\tilde{y}) = 1$, $C_{I_R(\tilde{x})}(\tilde{y}) = 1, C_{Q(\tilde{x})}(\tilde{y}) = 1$. Then we obtain:

$$\left(C_{I_L(\tilde{x})} \wedge C_{I_R(\tilde{x})} \wedge C_{Q(\tilde{x})}\right)(\tilde{y}) = \min\{C_{I_L(\tilde{x})}(\tilde{y}), C_{I_R(\tilde{x})}(\tilde{y}), C_{Q(\tilde{x})}(\tilde{y})\} = 1.$$

On the other hand, we have:

$$C_{I_L(\tilde{x})} \circledcirc C_{I_R(\tilde{x})} \circledcirc C_{Q(\tilde{x})} = C_{(I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]}.$$

Thus, $C_{(I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]}(\tilde{y}) = 1$, then $\tilde{y} \in (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]$. Based on Lemma 2.1, then $S(B)^1$ is both intra-regular and left weakly-regular. ■

Lemma 3. *Let $(S(B)^1, \circ, \leq)$ be an ordered bilinear form semigroup with identity element '1'. Then we have equivalent statements:*

i. *$S(B)^1$ is both intra-regular and left weakly regular*
ii. *$I_L \cap I_R \cap B \subseteq (I_L I_R B]$ for every bi-ideal $\mathcal{B}$, every left ideal I_L and every right ideal I_R of the semigroup $S(B)^1$*
iii. *$I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})]$ for every $\tilde{x} \in S(B)^1$.*

Proof:

$(i \Longrightarrow ii)$ In this case we have $S(B)^1$ is both intra-reguler and left weakly regular. Then we will prove that $I_L \cap I_R \cap \mathcal{B} \subseteq (I_L I_R \mathcal{B}]$ for every bi-ideal $\mathcal{B}$, every left ideal I_L and every right ideal I_R of the semigroup $S(B)^1$. First, for every $\tilde{x} \in I_L \cap I_R \cap \mathcal{B}$, we have $\tilde{x} \in I_L$, $\tilde{x} \in I_R$ and $\tilde{x} \in \mathcal{B}$. We must remember that $S(B)^1$ is intra-regular, then there exist $\tilde{y}, \tilde{z} \in S(B)^1$ such that $\tilde{x} \leq \tilde{y}\tilde{x}^2\tilde{z}$. On the other hand $S(B)^1$ is left weakly regular, so there exist $\tilde{u}, \tilde{v} \in S(B)^1$ such that $\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x}$. Hence, we get:

$$\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x} \leq \tilde{u}(\tilde{y}\tilde{x}\tilde{x}\tilde{z})\tilde{v}\tilde{x} = \left((\tilde{u}\tilde{y})\tilde{x}\right)(\tilde{x}(\tilde{z}\tilde{v})\tilde{x}) \in (S(B)^1 I_L)(I_R S(B)^1) \subseteq I_L I_R \mathcal{B} \subseteq (I_L I_R \mathcal{B}].$$

Finally we prove that $I_L \cap I_R \cap \mathcal{B} \subseteq (I_L I_R \mathcal{B}]$.

$(ii \Longrightarrow iii)$ The second, we must prove $I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]$ for every $\tilde{x} \in S(B)^1$ if it is known $I_L \cap I_R \cap \mathcal{B} \subseteq (I_L I_R \mathcal{B}]$ for every bi-ideal $\mathcal{B}$ every left ideal I_L and every right ideal I_R of the semigroup $S(B)^1$. Take an arbitrary element $\tilde{x} \in S(B)^1$, then we obtain $I_L(\tilde{x})$ is a left ideal generated by $\tilde{x}$, $I_R(\tilde{x})$ is a right ideal generated by $\tilde{x}$ and $\mathcal{B}(\tilde{x})$ is bi-ideal generated by $\tilde{x}$. On the other hand, we have $I_L \cap I_R \cap \mathcal{B} \subseteq (I_L I_R \mathcal{B}]$. Finally we have

$$I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})].$$

$(iii \Longrightarrow i)$ The third, we have $I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})]$ for every $\tilde{x} \in S(B)^1$. Then we will prove $S(B)^1$ is both intra-regular and left weakly regular. Take an arbitrary element $\tilde{x} \in S(B)^1$, then we have:

$$\tilde{x} \in I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}\mathcal{B}(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})] \subseteq (I_L(\tilde{x})I_R(\tilde{x})S] \subseteq (I_L(\tilde{x})I_R(\tilde{x})]$$
$$= \left(\left(S(B)^1\tilde{x}\right](\tilde{x}\, S(B)^1]\right]$$
$$= \left(\left(\left(S(B)^1\tilde{x}\right)(\tilde{x}\, S(B)^1)\right]\right] = \left(\left(S(B)^1\tilde{x}\right)(\tilde{x}\, S(B)^1)\right] = \left(S(B)^1\tilde{x}^2\, S(B)^1\right].$$

Hence, $S(B)^1$ is both intra-regular.

The other case, we have:

$$\begin{aligned}
&\tilde{x} \in I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}(\tilde{x}) \subseteq (I_L(\tilde{x}) I_R(\tilde{x}) \mathcal{B}(\tilde{x})] \\
&= \big((\, S(B)^1 \tilde{x}](\tilde{x}\, S(B)^1](\, S(B)^1 \tilde{x} \cap \tilde{x}\, S(B)^1]\big] \\
&\subseteq \big((\, S(B)^1 \tilde{x})(\tilde{x}\, S(B)^1)(\, S(B)^1 \tilde{x} \cap \tilde{x}\, S(B)^1]\big] \\
&= \big((\, S(B)^1 \tilde{x})(\tilde{x}\, S(B)^1)(\, S(B)^1 \tilde{x} \cap \tilde{x}\, S(B)^1]\big] \\
&= \big((\, S(B)^1 \tilde{x}^2\, S(B)^1)(\, S(B)^1 \tilde{x} \cap \tilde{x}\, S(B)^1]\big] \\
&\qquad = (S(B)^1 \tilde{x}^2 (S(B)^1)^2 \cap S(B)^1 \tilde{x}^2\, S(B)^1 \tilde{x}\, S(B)^1] \\
&\subseteq (S(B)^1 \tilde{x} S(B)^1 \tilde{x} \cap S(B)^1 \tilde{x} S(B)^1 \tilde{x} S(B)^1] \\
&\quad \subseteq (S(B)^1 \tilde{x} S(B)^1 \tilde{x}].
\end{aligned}$$

So, we obtain $S(B)^1$ left weakly regular. ■

Lemma 4. *Let $(S(B)^1, \circ, \leq)$ be an ordered bilinear form semigroup with identity element '1'. The semigroup $(S(B)^1, \circ, \leq)$ is both intra regular and left weakly regular if and only if for every fuzzy left ideal I_L^F, fuzzy right ideal I_R^F and fuzzy quasi-ideal $\mathcal{B}^F$ of $S(B)^1$, we obtain:*

$$I_L^F \wedge I_R^F \wedge \mathcal{B}^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc \mathcal{B}^F$$

Proof.

If we have I_L^F is a fuzzy left ideal, I_R^F is fuzzy right ideal and $\mathcal{B}^F$ is a fuzzy bi-ideal of $S(B)^1$, respectively, then we obtain:

$$(I_L^F \wedge I_R^F \wedge \mathcal{B}^F)(\tilde{x}) \leq (I_L^F \circledcirc I_R^F \circledcirc \mathcal{B}^F)(\tilde{x}), \text{ for all } \tilde{x} \in S(B)^1.$$

On the other hand we have $S(B)^1$ is intra-regular. Based on the definition of intra-regular, so there exist $\tilde{y}, \tilde{z} \in S(B)^1$ such that $\tilde{x} \leq \tilde{y}\tilde{x}^2\tilde{z}$. We have $S(B)^1$ is left weakly regular too. Then there exist $\tilde{u}, \tilde{v} \in S(B)^1$ such that $\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x}$. Based on these two conditions, then we have:

$$\tilde{x} \leq \tilde{u}\tilde{x}\tilde{v}\tilde{x} \leq \tilde{u}(\tilde{y}\tilde{x}^2\tilde{z})\tilde{v}\tilde{x} = ((\tilde{u}\tilde{y})\tilde{x})(\tilde{x}(\tilde{z}\tilde{v})\tilde{x}).$$

We have $((\tilde{u}\tilde{y})\tilde{x})(\tilde{x}(\tilde{z}\tilde{v})\tilde{x}) \in A_{\tilde{x}}$. This proves that $A_{\tilde{x}} \neq \emptyset$. Thus:

$$\begin{aligned}
(I_L^F \circledcirc I_R^F \circledcirc \mathcal{B}^F)(\tilde{x}) &= \underset{(\tilde{p},\tilde{q}) \in A_{\tilde{x}}}{\text{maks}} \{\min\{I_L^F(\tilde{p}), (I_R^F \circledcirc \mathcal{B}^F)(\tilde{q})\}\} \\
&\geq \min\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), (I_R^F \circledcirc \mathcal{B}^F)(\tilde{x}(\tilde{z}\tilde{v})\tilde{x})\} \\
&= \min\Big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), \underset{(\tilde{r},\tilde{s}) \in A_{(\tilde{x}(\tilde{z}\tilde{v})\tilde{x})}}{\text{maks}} \{\min\{I_R^F(\tilde{r}), \mathcal{B}^F(\tilde{s})\}\}\Big\} \\
&\geq \min\big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), \min\big\{I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big), \mathcal{B}^F(\tilde{x})\big\}\big\} \\
&= \min\big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big), \mathcal{B}^F(\tilde{x})\big\}.
\end{aligned}$$

We know that I_L^F is a fuzzy left ideal of $S(B)^1$, so we have $I_L^F((\tilde{u}\tilde{y})\tilde{x}) \geq I_L^F(\tilde{x})$. And we know that I_R^F is a fuzzy right ideal of $S(B)^1$, so we have $I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big) \geq I_R^F(\tilde{x})$. So, we have:

$$\min\big\{I_L^F((\tilde{u}\tilde{y})\tilde{x}), I_R^F\big(\tilde{x}(\tilde{z}\tilde{v})\big), \mathcal{B}^F(\tilde{x})\big\} \geq \min\{I_L^F(\tilde{x}), I_R^F(\tilde{x}), \mathcal{B}^F(\tilde{x})\} = (I_L^F \wedge I_R^F \wedge \mathcal{B}^F)(\tilde{x})$$

Conversely, we have $I_L^F \wedge I_R^F \wedge \mathcal{B}^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc \mathcal{B}^F$, for every fuzzy left ideal I_L^F, fuzzy right ideal I_R^F and fuzzy quasi-ideal $\mathcal{B}^F$ of $S(B)^1$. Then we must prove that $S(B)^1$ is both intra-regular and left weakly-regular.

Based on the previous Lemma 3.1, It is enough to prove that:

$$I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap B(\tilde{x}) \subseteq (I_L(\tilde{x}) I_R(\tilde{x}) \mathcal{B}(\tilde{x})], \text{ for every } \tilde{x} \in S(B)^1.$$

First, we take an arbitrary element $\tilde{x} \in S(B)^1$ and $\tilde{y} \in I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap \mathcal{B}(\tilde{x})$. So, we have $\tilde{y} \in (I_L(\tilde{x}) I_R(\tilde{x}) \mathcal{B}(\tilde{x})]$. We know that if I_L^F, I_R^F and $\mathcal{B}^F$ are fuzzy left ideal, fuzzy right ideal and fuzzy bi-ideal, respectively, then $I_L(\tilde{x}), I_R(\tilde{x}), \mathcal{B}(\tilde{x})$ are left ideal, right ideal and bi-ideal generated by $\tilde{x}$, respectively. So the characteristic functions $C_{I_L(\tilde{x})}$, $C_{I_R(\tilde{x})}$, $C_{\mathcal{B}(\tilde{x})}$ are fuzzy left ideal, fuzzy right ideal and fuzzy quasi ideal of $S(B)^1$, respectively. Based on the hypothesis, we have:

$$\big(C_{I_L(\tilde{x})} \wedge C_{I_R(\tilde{x})} \wedge C_{\mathcal{B}(\tilde{x})}\big)(y) \leq \big(C_{I_L(\tilde{x})} \circledcirc C_{I_R(\tilde{x})} \circledcirc C_{\mathcal{B}(\tilde{x})}\big)(\tilde{y}).$$

We know that $\left(C_{I_L(\tilde{x})} \wedge C_{I_R(\tilde{x})} \wedge C_{\mathcal{B}(\tilde{x})}\right)(\tilde{y}) = \min\{C_{I_L(\tilde{x})}(\tilde{y}), C_{I_R(\tilde{x})}(\tilde{y}), C_{\mathcal{B}(\tilde{x})}(\tilde{y})\}$. So, we get $\min\{C_{I_L(\tilde{x})}(\tilde{y}), C_{I_R(\tilde{x})}(\tilde{y}), C_{\mathcal{B}(\tilde{x})}(\tilde{y})\} \leq \left(C_{I_L(\tilde{x})} \circledcirc C_{I_R(\tilde{x})} \circledcirc C_{\mathcal{B}(\tilde{x})}\right)(\tilde{y})$. Since $\tilde{y} \in I_L(\tilde{x})$, $\tilde{y} \in I_R(\tilde{x})$, and $\tilde{y} \in \mathcal{B}(\tilde{x})$, so $C_{I_L(\tilde{x})}(\tilde{y}) = 1$, $C_{I_R(\tilde{x})}(\tilde{y}) = 1, C_{Q(\tilde{x})}(\tilde{y}) = 1$. Then we obtain:

$$\left(C_{I_L(\tilde{x})} \wedge C_{I_R(\tilde{x})} \wedge C_{\mathcal{B}(\tilde{x})}\right)(\tilde{y}) = \min\{C_{I_L(\tilde{x})}(\tilde{y}), C_{I_R(\tilde{x})}(\tilde{y}), C_{\mathcal{B}(\tilde{x})}(\tilde{y})\} = 1.$$

On the other hand, we have:

$$C_{I_L(\tilde{x})} \circledcirc C_{I_R(\tilde{x})} \circledcirc C_{\mathcal{B}(\tilde{x})} = C_{(I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})]}.$$

Thus, $C_{(I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})]}(\tilde{y}) = 1$, then $\tilde{y} \in (I_L(\tilde{x})I_R(\tilde{x})\mathcal{B}(\tilde{x})]$. Based on the definition then $S(B)^1$ is both intra-regular and left weakly regular. ■

CONCLUSIONS

Based on the previous discussion we conclude that there are same properties between fuzzy quasi-ideal and fuzzy bi-ideal of an ordered bilinear form semigroup $S(B)^1$. These properties are:

1. If $(S(B)^1, \circ, \leq)$ is an ordered bilinear form semigroup with identity element '1', then we have equivalent statements:
   i. $S(B)^1$ is both intra-regular and left weakly regular
   ii. $I_L \cap I_R \cap Q \subseteq (I_L I_R Q]$ for every quasi-ideal Q every left ideal I_L and every right ideal I_R of the semigroup $S(B)^1$
   iii. $I_L(\tilde{x}) \cap I_R(\tilde{x}) \cap Q(\tilde{x}) \subseteq (I_L(\tilde{x})I_R(\tilde{x})Q(\tilde{x})]$ for every $\tilde{x} \in S(B)^1$.

   By substitute the quasi ideal with bi-ideal we will get the similar property.
2. If $(S(B)^1, \circ, \leq)$ be an ordered bilinear form semigroup with identity element '1', then semigroup $(S(B)^1, \circ, \leq)$ is both intra regular and left weakly regular if and only if for every fuzzy left ideal I_L^F, fuzzy right ideal I_R^F and fuzzy quasi-ideal Q^F of $S(B)^1$, we obtain:

$$I_L^F \wedge I_R^F \wedge Q^F \preccurlyeq I_L^F \circledcirc I_R^F \circledcirc Q^F$$

ACKNOWLEDGMENTS

This research is supported by Fundamental Research Grant provided by Dirjen DIKTI.

REFERENCES

1. Asaad,M.: Group and Fuzzy Subgroup. *Fuzzy Sets and systems* 39 , 1999, pp: 323 - 328.
2. Howie, J.M., *An Introduction to Semigroup Theory*. Academic Press. London, 1976.
3. Karyati. :Semigrup yang Dikonstruksikan dari Bentuk Bilinear. *Tesis*, Program Pascasarjana Universitas Gadjah Mad. Yogyakarta, 2002.
4. Karyati, Wahyuni, S., The Properties of Non-degenerate Bilinear Form. *Proceeding of SEAMS-GMU: International Conference on Mathematics and Its Applications*, 2003.
5. Karyati, Wahyuni, S, Surodjo, B, Setiadji., Quotient Semigroups Induced by Fuzzy Congruence Relations, *Proceeding IndoMS International Conference on Mathematics and Its Application (IICMA)*, GMU, Yogyakarta, 2009, pp: 102-111.
6. Karyati, Wahyuni, S, Surodjo, B, Setiadji., The Fuzzy Regularity of Bilinear Form Semigroups, *Proceedings of "The 6th SEAMS-UGM Conference 2011"* , 2012.
7. Karyati, Wahyuni, S, Surodjo, B, Setiadji., Membangun Suatu Relasi Fuzzy pada Semigrup Bentuk Bilinear. *Prosiding Seminar Nasional Jurusan Matematika*, Universitas Sebelas Maret., 2013.
8. Karyati, Dhoriva, U.W., Semigrup Bentuk Bilinear Terurut Parsial dalam Batasan Subhimpunan Fuzzy. *Seminar Nasional Matematika dan Pendidikan Matematika*, PPs Universitas Sebelas Maret . 2013.
9. Kehayopulu, N., Ideals and Green Relations in Partial ordered Semigroups, *International Journal of Mathematics and Mathematical Sciences*, Volume 26, 2005, pp: 1-8.
10. Kehayopulu, N., Left Regular Partial ordered Semigroups in which the Fuzzy Left Ideals are Two-Sided, *International Journal of Algebra*, Vol 6, no.10, 2012, pp:493-499.

11. Klir, G.J, Clair, U.S, Yuan, B., *Fuzzy Set Theory: Foundation and Applications*. Prentice-Hall, Inc. USA. 1997.
12. Mohanraj, G, Krishnaswamy, D and Hema, R., On Generalized Redefined Fuzzy Prime Ideals of Partial ordered Semigroups, *Annals of Fuzzy Mathematics and Informatics*, Volume X, No 10, 2011, pp: 1- 9.
13. Mordeson, J.N, Malik, D.S., Fuzzy Commutative Algebra. World Scientifics Publishing Co. Pte. Ltd. Singapore, 1998.
14. Murali, V., Fuzzy Equivalence Relation. *Fuzzy Sets and System* 30, pp: 155-163. (1998)
15. Rajendran, D, Nambooripad, K.S.S., Bilinear Form and a Semigroup of Linear Transformations. *Southeast Asian Bulletin of Mathematics* 24, 2000, p: 609-616 .
16. Shabir, M, Khan, A., Characterizations of Partial ordered Semigroups by the Properties of Their Fuzzy Ideals, *Computers and Mathematics with Applications*, Volume 59, 2010, pp: 539 – 549.

Proceedings of "The 7^{th} SEAMS-UGM Conference 2015"
Topic, pp. 95–108.

Study of Two Fluid Flows Model by Using the MPS Method

Mario Batubara* and Suprijadi Harjono†

*Space Science Center of Indonesian National Institute of Aeronautics and Space,
Email: mario.batubara@lapan.go.id
†Bandung Institute of Technology

Abstract. The Moving Particle Semi-implicit (MPS) Methods is a commonly method for simulating fluid flow. Most of the simulations that has been generated by the MPS method is a single fluid flow simulation. At first, numerically, the MPS method is not stable enough to simulate the fluid flows and physical reactions. Along with its development, MPS has increased in terms of numerical stability, conservation of momentum and mechanical energy calculations. On the other hand, the method is very depend on the fluid density and viscosity. The continuity of the process flow of fluid is also greatly influenced by the value of the comparison of each of these parameters. This research aims to study the two fluid flows model by using the MPS method. The expected outcome of this study is to obtain the relationship between the value of the ratio of these two parameters to the equilibrium time achievement due to the total kinetic energy of the fluid.

Keywords: Moving Particle Semi-implicit (MPS), two fluid, density, viscosity.

INTRODUCTION

Background

Interactions between particles in a fluid flow (e.q. liquid and gasses) play an important role in the natural phenomenon that occurs in the environment. In addition, the two-fluid interaction process is widely applied in various industries and research for safety and environmental sustainability. This interaction process can be analyzed based on fluid dynamics solutions which is obtained from some computation fluid properties such as velocity flow, pressure, density, and temperature, as a function of space and time. The calculations of fluid properties use the equations referring to the principle of conservation laws, specifically, conservation of mass, conservation of linear momentum and conservation of energy. The equation is the Navier-Stokes equation which is a form of momentum equation which is depends on the velocity and pressure gradient. Solving these equations can be done by several simplification methods of computation in the Computational Fluid Dynamics. One of the methods of computational fluid dynamics is Moving Particle Semi-implicit (MPS) which has its own advantages compared to other methods. MPS method has advantages in free surface flow of incompressible fluid, able to solve problems in the interaction between the particles of the fluid-rigid, and water droplets fall problems. Over time, the MPS method has been widely applied in the field of engineering applications include Nuclear Engineering in [7] and [9], Coastal Engineering in [4] and [5], Environmental Hydraulics in [13], Ocean Engineering in [18] and [19], Structural Engineering in [3], Mechanical Engineering in [6] and [21], Bioengineering in [22], and Chemical Engineering in [20].

Main Purposes

In this paper, we explain some results which is related to the study of two fluid flows model by using the MPS method. As the result, we shows the relationship between ratio of density and viscosity of fluid to the equilibrium time achievement due to the total kinetic energy of the fluid. We use a reliable combination value of fluid density and viscosity which describe an incompressible fluid parameter to obtain the results above.

FLUID DYNAMICS MODELLING

The movement of fluid particles and fluid dynamics are governed by Navier-Stokes equations in the form of differential equations. The equation is derived from the principles of conservation and Newton's second law in which the forces acting on the fluid (internal forces and external forces) can affect the acceleration of each fluid particle. On the other hand, the mass conservation is also taken into calculation in fluid dynamics modelling.

Continuity of Mass

In other words, it is called conservation of mass where the rate of change of fluid mass inside a control volume must be equal to the net rate of fluid flow into the volume. This statement can be translated into the differential form of the continuity equation as written on equation 1:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{1}$$

where ρ is fluid density, $\vec{u}$ is the fluid velocity vector. The second term of equation 1 is material derivative describing the density change of physical quantity that can be affected by displacement and movement along with the advection in the form $\vec{u} \cdot \nabla \rho$. In the case of incompressible flow where the control volume is kept constant (isochoric flow) and fluid density did not change ($\frac{D\rho}{Dt} = 0$), the mass continuity equation simplifies to a volume continuity equation $\nabla \cdot \vec{u} = 0$ which describes the velocity of fluid flow has the same value in both direction, inflow and outflow of the control volume.

Momentum Conservation

The momentum of a particle can be obtained by multiplying the mass of the particle and its velocity. The total of each particle momentum is constant, so that the momentum can not be created nor destroyed, but momentum can only be changed through a total force $\vec{F}$ and pressure gradient force $\frac{1}{\rho}\nabla P$ acting on the particles as described by Newton's second law of motion of an object. The differential form of the momentum conservation equation is written on equation 2 :

$$\frac{D\vec{u}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla P \tag{2}$$

The term of total force in the case of the Navier-Stokes equations is a body force which is in term of diffusion and the external source (e.q. gravitational force).

Particle Method for Fluid Dynamic Modelling

The simulation method based on particle interaction has become a solution to solve the complex fluid flow problems that occur when using a grid-based method. In particle method, differential operators for the mass and momentum conservation equations need to be transformed by the corresponding particle interaction operator. In other words, partial differential equations in the Navier-Stokes system in the context of continuous interaction equation is converted into discrete particles. Therefore, the transport equation can be approximated based on the particles interaction. The transformation of particles interaction operator is depend on the kernel function which is correlate to an effective distance between particles. Kernel functions are illustrated by Shakibaeinia and Jin [13, 14, 15, 16] shown in Figure 1. A particle i located in the center of kernel area within radius r_e interact with surrounding particles which have an effective distance r_j that can determine the value of the kernel function $w(r_{ij})$, where $r_{ij} = |r_j - r_i|$. The larger an effective range of kernel function, the more particles interact. So far, there are some kernel functions are developed (Table 1) and summarized by Ashtiani and Farhadi [1], where the first kernel function [KF1] shows exponential characteristically, the second kernel function [KF2] characterized by cubic spline, The third kernel function [KF3] characterized by fourth order spline function and the fifth kernel function which proposed by Shao and Lo in 2003 has the characteristic of cubic spline. Additionally, Koshizuka et al. [10] developed the kernel function [KF4] that have different characteristics from the previous fourth kernel functions.

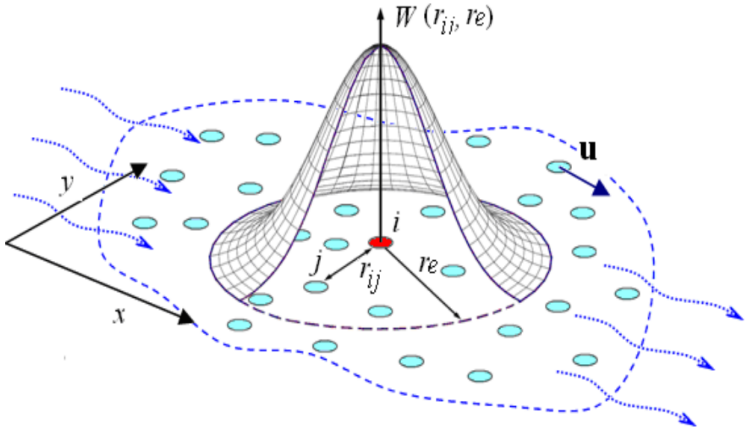


FIGURE 1. Weighting Function Scheme

TABLE 1. Table of Kernel Function

No.	Kernel Function	Reference
KF1	$w(r)=\begin{cases} e^{-(r/\alpha r_e)^2} & \text{if } 0 \leqslant r \leqslant r_e \\ 0 & \text{if } r_e < r \end{cases}$	[2]
KF2	$w(r)=\begin{cases} \frac{2}{3}-4\left(\frac{r}{r_e}\right)^2+4\left(\frac{r}{r_e}\right)^3 & \text{if } 0 \leqslant r \leqslant \frac{r_e}{2} \\ \frac{4}{3}-4\left(\frac{r}{r_e}\right)+4\left(\frac{r}{r_e}\right)^2-\frac{4}{3}\left(\frac{r}{r_e}\right)^3 & \text{if } \frac{r_e}{2} \leqslant r \leqslant r_e \\ 0 & \text{if } r_e < r \end{cases}$	[2]
KF3	$w(r)=\begin{cases} 1-6\left(\frac{r}{r_e}\right)^2+8\left(\frac{r}{r_e}\right)^3-3\left(\frac{r}{r_e}\right)^4 & \text{if } 0 \leqslant r \leqslant r_e \\ 0 & \text{if } r_e < r \end{cases}$	[2]
KF4	$w(r)=\begin{cases} \frac{r_e}{r}-1 & \text{if } 0 \leqslant r < r_e \\ 0 & \text{if } r_e \leqslant r \end{cases}$	[10]
KF5	$w(r)=\begin{cases} \frac{40}{7\pi r_e^2}\left(1-6\left(\frac{r}{r_e}\right)^2+6\left(\frac{r}{r_e}\right)^3\right) & \text{if } 0 \leqslant r < 0.5r_e \\ \frac{10}{7\pi r_e^2}\left(2-2\frac{r}{r_e}\right)^3 & \text{if } 0.5r_e < r < r_e \\ 0 & \text{if } r > r_e \end{cases}$	[17]

The fifth graph kernel function based on the table 1 can be seen in Figure 2. All functions except KF4 has an infinity value at r = 0. While KF1, KF2, KF3 and KF5 have the same characteristics. Based on the fourth kernel function above, KF5 is closer to the characteristic of delta function, where it shows more normally distributed than four other kernel functions. All graphs use the same condition criteria. The effective radius of particle should be larger than particle radius.

INTRODUCTION OF MPS METHOD

Computation Method in MPS

There is another method beside MPS which can be applied into fluid dynamics analysis. The method is called Smoothed Particle Hydrodynamic (SPH) whose founded by Gingold and Monaghan in 1977. Both method, MPS and SPH provides approximations to the partial differential equations based on the integral interpolation. However, the

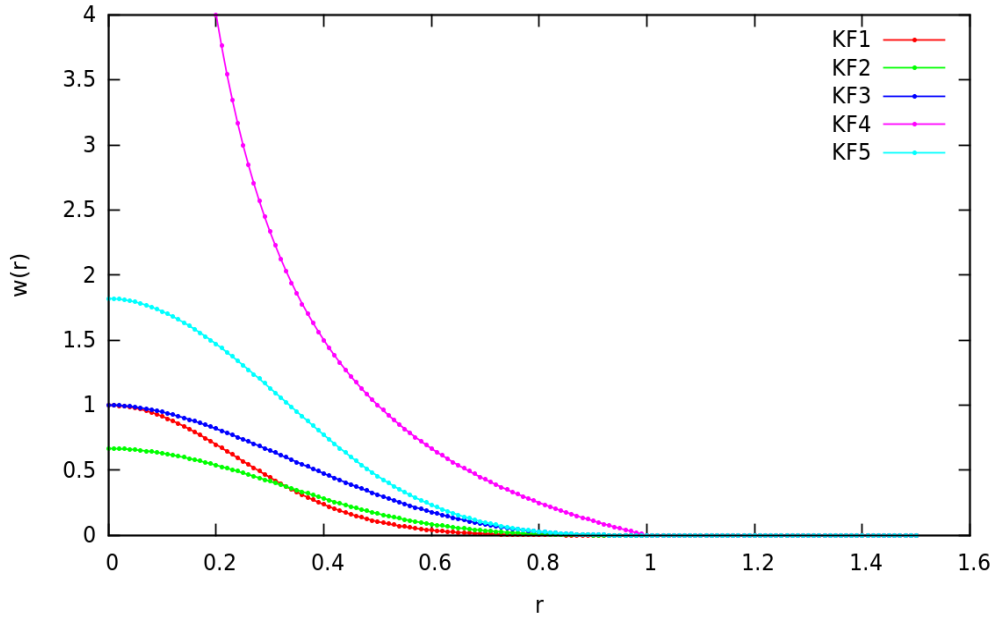


FIGURE 2. The Fifth Kernel Function

solution process of the MPS method differs to the SPH method as the solution to the partial differential equations which is applies simplified differential operator model based on the local weighted averaging of a kernel function and obtained through a semi-implicit prediction-correction process rather than in the original SPH method with fully explicitly. Koshizuka in [8], [10] and [9] founds some approximate weighting function for fluid particle parameters (particle number density n (equation 3), pressure gradient ∇P (equation 4), and the Laplacian form for the velocity $\nabla^2 \vec{u}$ (equation 5).

$$\langle n \rangle_i = \sum_{j \neq i} w\left(\left|\mathbf{r}_j - \mathbf{r}_i\right|\right) \tag{3}$$

$$\langle \nabla P \rangle_i = \frac{d}{n^0} \sum_{j \neq i} \left[\frac{P_j - P_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^2} \left(\mathbf{r}_j - \mathbf{r}_i\right) w\left(\left|\mathbf{r}_j - \mathbf{r}_i\right|\right) \right] \tag{4}$$

$$\left\langle \nabla^2 \vec{u} \right\rangle_i = \frac{2d}{\lambda n^0} \sum_{j \neq i} \left(\vec{u}_j - \vec{u}_i\right) w\left(\left|\mathbf{r}_j - \mathbf{r}_i\right|\right) \tag{5}$$

Furthermore equation 6, showing the approximating value of the number of particles in unit volume ρ_n that can be determined by using the particle number density [8].

$$\langle \rho_n \rangle_i = \frac{\langle n \rangle_i}{\int w\left(\left|\mathbf{r}_j - \mathbf{r}_i\right|\right) dv}, \tag{6}$$

On the other hand, the number of particles with different densities is calculated simultaneously, the mass m will change with the particle number density n^0 and it turned into a temporary particle number density n^*. Therefore, the particle number density n^0 obtained based on the sum ofthe n^* and n' as written in equation 7, [8].

$$n^* + n' = n^0, \tag{7}$$

where n' is the correction of particle number density. The new form of velocity divergent (equation 8) is modified by substitute equation 7 into equation 1.

$$\frac{1}{\Delta t} \frac{n'}{n^0} = -\nabla \cdot \vec{u}', \tag{8}$$

Implicitly, the correction value of velocity $\vec{u}'$ can be derived by using the form of pressure gradient ∇P on the conservation of momentum equation 2. Therefore, the correction value of velocity can be calculated by using equation 14, [8].

Based on the results written in the equation 14, the complete form of the pressure gradient (equation 14) can be tranformed into the complete form of pressure gradient shown in the equation 9, [8].

$$\frac{d}{n^0}\sum_{j\neq i}\left[\frac{P_j-P_i}{\left|\mathbf{r}_j-\mathbf{r}_i\right|^2}\left(\mathbf{r}_j-\mathbf{r}_i\right)w\left(\left|\mathbf{r}_j-\mathbf{r}_i\right|\right)\right]=-\frac{\rho}{\Delta t}\vec{u}', \tag{9}$$

where P_i on the equation 4 and 9 was changed into another form of $\hat{P}_i$ to maintain numerical stability according to the condition as shown in the equation 10, [8]. The minimum value has been chosen to avoid the negative reflection forces between particle caused by $P_j-\hat{P}_i$ where the pressure here is obtained from the solution of Laplace equation in the form of pressure (equation 11), [8].

$$\hat{P}_i=\min_{j\in J}\left(P_i,P_j\right), J=\left\{j:w\left(\left|\mathbf{r}_j-\mathbf{r}_i\right|\right)\neq 0\right\} \tag{10}$$

$$\left\langle\nabla^2 P^{n+1}\right\rangle_i=-\frac{\rho}{\Delta t^2}\frac{\langle n^*\rangle_i-n^0}{n^0}, \tag{11}$$

Discretization form of poison equation written in Equation 11 can be transformed into a linear equation system (linearization)and it can be solved by several methods such as iterative and direct method. In this study, a solution Poisson equation obtained by using LU Decomposition techniques as the direct method. This method is based on the triangularization technique, which aims to eliminate the unknown components systematically to the system of equations $\mathrm{A}\mathbf{x}=\mathbf{b}$. In addition, the matrix A is the lower-upper-triangular matrix(A = LU), where the value of matrix element $A_{mk}=A_{km}=\frac{2d}{\lambda n^0}w\left(\left|\mathbf{r}_j-\mathbf{r}_i\right|\right)$, matrix $A_{mm}=\frac{2d}{\lambda n^0}\sum_{j\neq i}-w\left(\left|\mathbf{r}_j-\mathbf{r}_i\right|\right)$, matrix $b_i=-\frac{\rho}{\Delta t^2}\frac{\langle n^*\rangle_i-n^0}{n^0}$. Therefore, the equations $\mathrm{A}\mathbf{x}=\mathbf{b}$ is equivalent with $\mathrm{LU}\mathbf{x}=\mathbf{b}$, where the matrix LU split into two triangular matrix form $\mathrm{L}\mathbf{y}=\mathbf{b}$ that can be solved by forward-substitution technique (equation 12) to get the solution $\mathbf{y}_i$.

$$\mathbf{y}_i=\frac{1}{L_{ii}}\left[\mathbf{b}_i-\sum_{k=1}^{i-1}L_{ik}\mathbf{y}_k\right], i=1,2,\cdots,n \tag{12}$$

Another term of matrix LU is $\mathrm{U}\mathbf{x}=\mathbf{y}$ that can be solved by back-substitution technique (equation 13) to get the solution $\mathbf{x}_i$.

$$\mathbf{x}_i=\frac{1}{U_{ii}}\left[\mathbf{y}_i-\sum_{k=i+1}^{n}U_{i,k}\mathbf{x}_k\right], i=n,n-1,\cdots,1 \tag{13}$$

$$\begin{aligned}\frac{D\vec{u}}{Dt}&=-\frac{1}{\rho}\nabla P^{n+1},\\ \vec{u}'&=-\frac{\Delta t}{\rho}\nabla P^{n+1},\end{aligned} \tag{14}$$

Algorithm of MPS Method

The main keywords in the MPS method are explicit and implicit where the velocity and position of a particle is predicted explicitly in the form of non-pressure term involving the viscous force and other external forces to get

the predicted value of the speed and position of a particle $\mathbf{r}^*$ dan $\vec{u}^*$. The incompressible flow where the density of the number of particle density should be kept constant by correcting the prediction value of particle velocity which is obtained previously. The Correction of particle velocity can be calculated by the pressure gradient through the pressure Poisson equation implicitly. So that, the semi-implicit is composed of prediction and correction where a correction speed should be done implicitly by using predictive calculation in explicit process. Mechanism techniques employed in the MPS method showed on table 2.

TABLE 2. Algorithm of MPS

Algorithm of MPS

initialize all the attributes of fluid particles
initialize all the attributes of image output
calculate the initial number density of particles
count the number of neighbors of each particle
calculate λ
specify particle boundary and particle interface
while simulation time is not over ***do***
  for each particle ***do***
    EXPLICIT PHASE.
      count the whole force **f** acting on each particle.
      calculate the speed and temporary position of the entire particle: $\mathbf{u}^*$ dan $\mathbf{r}^*$.
      $$\mathbf{u}^* = \mathbf{u}^n + \mathbf{f}\Delta t.$$
      $$\mathbf{r}^* = \mathbf{r}^n + \mathbf{u}^*\Delta t.$$
      calculate the number density of each particle n^* based on $\mathbf{r}^*$.
      calculate the solution of pressure term on the Poisson Eqn.
      using LU_Decomposition.
      $$\nabla^2 P^{n+1} = -\frac{\rho}{\Delta t^2}\frac{n^* - n^0}{n^0}.$$
    IMPLICIT PHASE.
      calculate pressure gradient of each particle.
      calculate velocity correction of the particle **u**'.
      $$\mathbf{u}' = -\frac{\Delta t}{\rho}\nabla P^{n+1}.$$
    UPDATING VELOCITY DAN PARTICLE POSITION.
      $\mathbf{u}^{n+1} = \mathbf{u}^* + \mathbf{u}'$.
      $\mathbf{r}^{n+1} = \mathbf{r}^* + \mathbf{u}^{n+1}\Delta t$.
  end for
    GNUPLOT PROCEDURE.
      profile of velocity, position, and pressure of the particle.
end while

surface tension model

Model surface tension used in this paper is a model of the surface tension built and developed by Nomura et al. in [12]. The model shown in the following equation 15:

$$\vec{F}_{st} = \sigma\kappa\delta\hat{n}, \tag{15}$$

where σ, κ, δ, and $\hat{n}$ respectively are surface tension coefficient, *curvature*, delta function and normal vector.

MODELLING TWO FLUID FLOW USING MPS METHOD

In this section, we will discuss more about the modelling of multi fluid which is effected by the density and viscosity difference of the fluid. In addition, differential operator also modified for multi fluid flow problems such as pressure gradients, surface tension, buoyancy, viscous force, as well as boundary conditions and particle search techniques will added for complementary discussion of this chapter.

pressure gradient model

The fluid density take important effect in the fluid flow problem. The density itself give the proportional portion into the buoyancy force which is not involving in the single fluid problem. However, in the multi fluid flow problems, especially the flow of two fluids with different density will affect the buoyant force of a fluid. The effect of this buoyant force is combined with the pressure gradient models (equation 9). Therefore, the form $\vec{u}'$ can be written as shown in the following equation 16, [11].

$$\begin{aligned} \vec{u}' = -\frac{\Delta t}{\rho}\frac{d}{n^0}\sum_{j\neq i}\left[\frac{P_j - P_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^2}\left(\mathbf{r}_j - \mathbf{r}_i\right) w\left(\left|\mathbf{r}_j - \mathbf{r}_i\right|\right)\right] - \vec{u}_i^B, \\ \text{where } \vec{u}_i^B = \sum_{j\neq i}\mathbf{g}\left(\rho_j - \rho_i\right) f_B c_B w\left(\left|\mathbf{r}_j - \mathbf{r}_i\right|\right), \end{aligned} \tag{16}$$

where $\vec{u}_i^B$ is velocity influenced by buoyant force effect, c_B is constant of buoyant force effect, f_B is factor of individual particles related to particle distance.

viscous force

Another important factor in fluid flow problem is fluid viscosity which is reflect to the viscous force . Back to equation 2, it is clear that the force also need to be taken into the calculation process when analyzing multi fluid flow problem. Shakibaenia and Jin [16] defines a viscous factor approximation (equation 17) to analyze multiple fluid flow problem,

$$\langle \nu \rangle_{ij} \approx \frac{\nu_i \nu_j}{\nu_i + \nu_j} \tag{17}$$

where $\langle \nu \rangle_{ij}$ is an approximation of individual particle viscosity. The original viscous force in term of Laplacian of the velocity can be expand into the equation 18 by substitute a viscosity approximation of individual particle above.

$$\begin{aligned} \left\langle \nu \nabla^2 \vec{u} \right\rangle_i &\approx \frac{2d}{\lambda n^0}\sum_{j\neq i}\nu_{ij}(\vec{u}_j - \vec{u}_i) w(\left|\mathbf{r}_j - \mathbf{r}_i\right|) \\ &\approx \frac{2d}{\lambda n^0}\sum_{j\neq i}\frac{\nu_i \nu_j}{\nu_i + \nu_j}(\vec{u}_j - \vec{u}_i) w(\left|\mathbf{r}_j - \mathbf{r}_i\right|) \end{aligned} \tag{18}$$

System Configuration and Results

Figure 3 shows our system configuration design in detail. Some parameters as particle positions, pressure and velocity of each particle should be determined as initialization value. The dimension of box also should be determined as position and its category. Initial parameters and calculation parameters used in the simulation are listed in Table 3 and 4.

TABLE 3. Initial Parameters.

Parameter			
Particle Number	N	2790	–
Initial Viscousity 1	μ_i^1	10^{-6}	kgm/s
Initial Viscousity 2	μ_i^2	10^{-3}	kgm/s

Figure 4 shows some simulations which is resulted by governing the two fluid flows model described in the previous chapter. The figure displays the fluid particle parameters such as position, velocity and pressure temporary until it reaches equilibrium and the results based on the comparative of density $\frac{\rho_d}{\rho_f} = 1.4$. Figure 5 and 6 also shows multiple views of particle pressure and velocity profiles during the simulation. Pressure profile already showing the degradation contour vertically which could explain the hydrostatic pressure of the fluid.

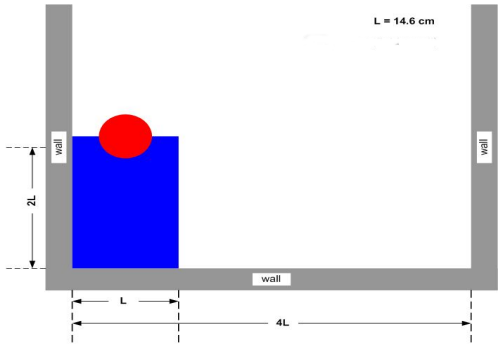


FIGURE 3. System Configuration

TABLE 4. Parameters of Calculation

Kernel Size			
Kernel distance for particle density estimation	r_{en}	$2.1l_0$	–
Kernel distance for Laplace calculation	r_{eLap}	$4.0l_0$	–
Time step			
Maximum of Δt		$\Delta t \leq 10^{-3}$	s
Free surface			
Free surface parameter 1	β_1	0.97	–
Free surface parameter 2	β_2	0.97	–
Particle distance			
Initial particle distance	l_0	8.0×10^{-3}	m

For further analysis, some simulation experiments has been done to study the effect of density ratio of two fluids. The experiment also use the same simulation period which will stop until the equilibrium conditions reached. As the result, figure 8 shows the trend line for the comparison of the effect of the fluid density against time achievements until equilibrium conditions reached. The trend shown in Figure 8 due to differences in particle density fluid that gives the effect of buoyant force on the particles. Fluid particles which have a lower density will have a greater buoyancy than the particles having a low density value. Other than that, the effect of viscosity differences are also explain its influence on the achievement time to reach equilibrium conditions. Figure 7 shows a graph showing the effect. It shown clear that when $\nu_d < \nu_f$ took longer to reach equilibrium rather than $\nu_d > \nu_f$. Slope of the curve also shows the differences in both conditions $\frac{\nu_d}{\nu_f} < 1$ and $\frac{\nu_d}{\nu_f} > 1$ where the first conditions is steeper than the slope of the curve on the other one. Therefore, it shows that the fluid viscosity plays roles in the ability of fluid flow.

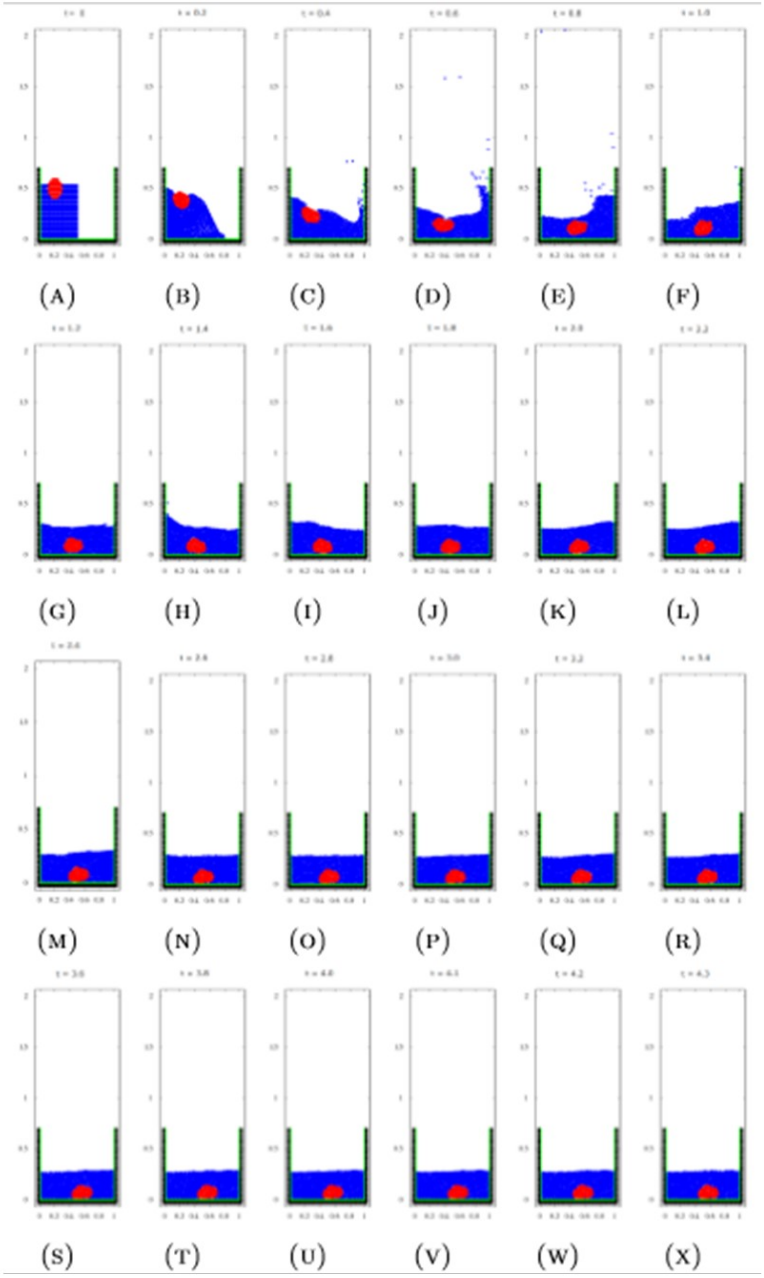


FIGURE 4. Screen Shoot fluid particle position **r**

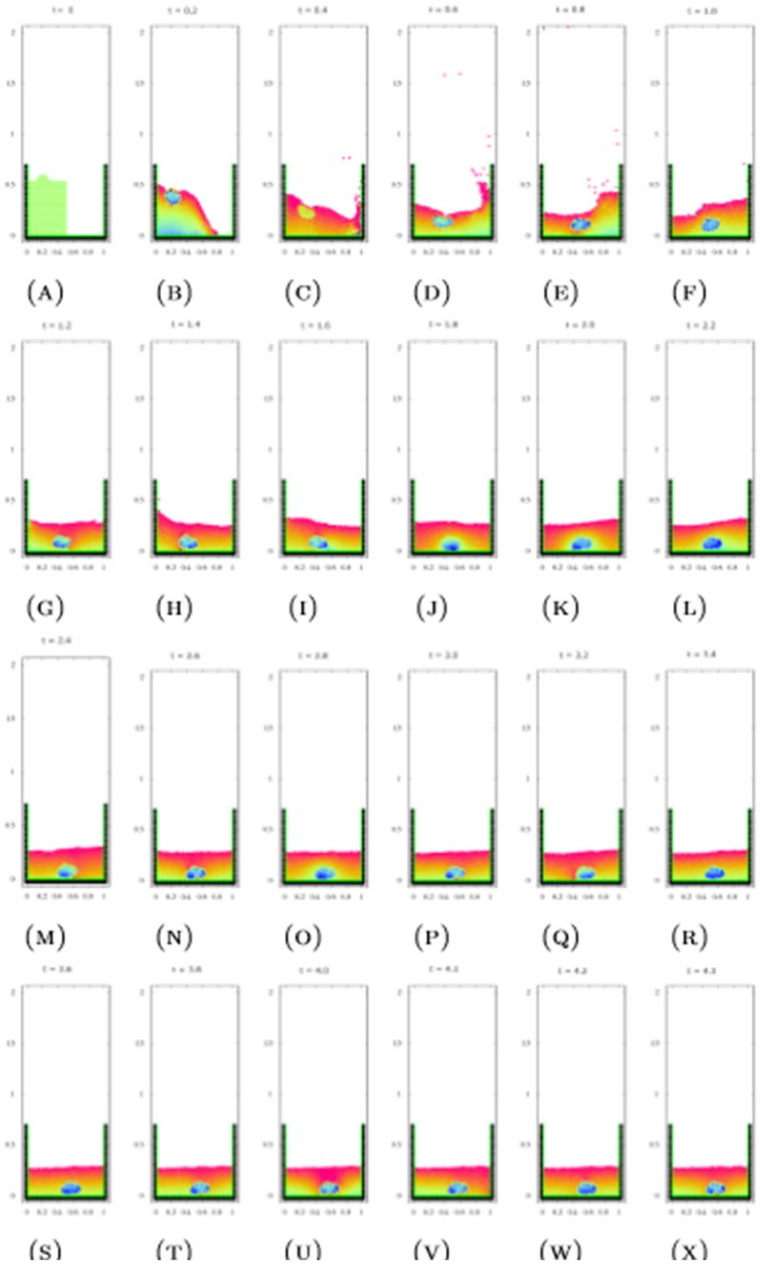


FIGURE 5. Screen Shoot fluid particle pressure

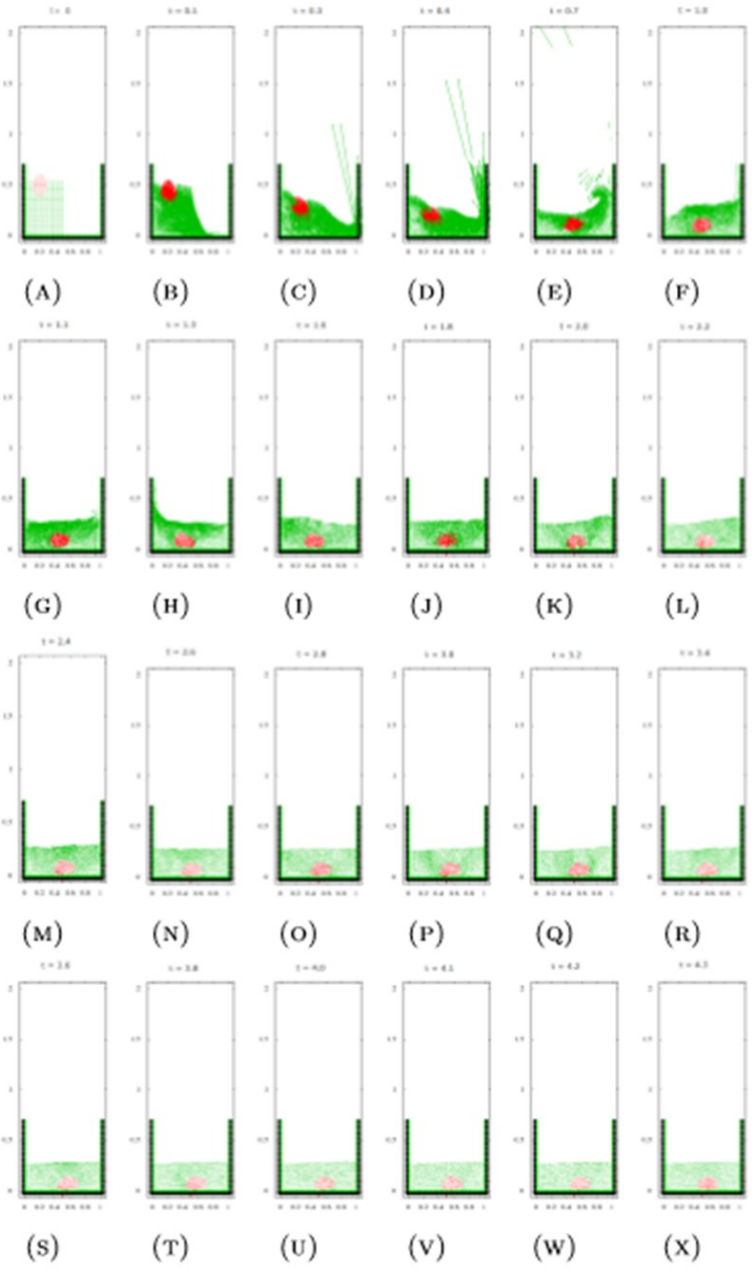


FIGURE 6. Screen Shoot fluid particle velocity

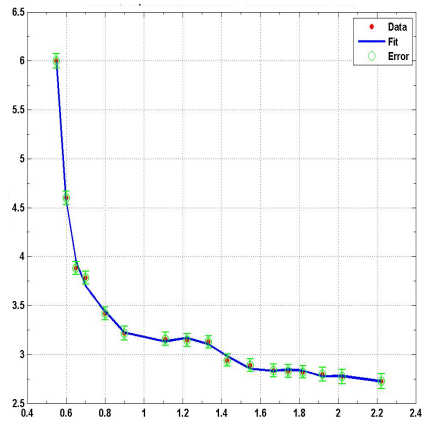


FIGURE 7. Rate of Two Fluid Viscosity Effect to the Time Achievement of Equilibrate Conditions [$\frac{\nu_d}{\nu_f}$ vs $t_{equilib}$]

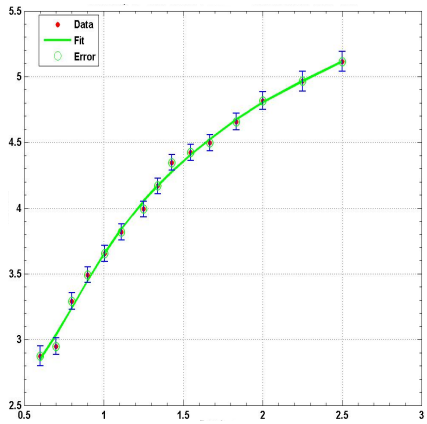


FIGURE 8. Rate of Two Fluid Density Effect to the Time Achievement of Equilibrate Conditions [$\frac{\rho_d}{\rho_f}$ vs $t_{equilib}$]

CONCLUDING REMARKS

Based on the discussion and results above, it can be state as a conclude remarks where the MPS method also can be implement into the two fluid flow problems. A simple dam break simulation has been done for studying the two fluid flows model based on the MPS method. Larger particle density and determination of fluid interface shape are interesting to study especially on the effect of the fluid density and viscosity ratio to the total kinetic energy. The different of both fluid parameter ratio especially in the ratio of fluid density. The fluid droplets whose $\rho_d < \rho_f$ is under the influence of the buoyant force is greater than the opposite situation, the equilibrium condition takes more quickly. Different case on the viscosity effect, when the viscosity of droplet is lower than the viscosity of fluids at rest, the droplet particles take longer time to reach the equilibrium conditions than when the viscosity of the droplet is greater than the viscosity of the fluid. The peak time which is required to reach equilibrium condition occurs when both fluid viscosity value are equal ($\nu_d = \nu_f$).

ACKNOWLEDGMENTS

We wish to give a special thank to the Indonesian Institute of Aeronautics and Space for funding this works. Another wish we thank to the Department of Physics, Faculty of Mathematics and Natural Science, ITB which has give so many facility especially in conducting computer facility.

REFERENCES

1. Ashtiani, B. A. and Farhadi, L., A Stable Moving-Particle Semi-implicit Method for free Surface Flows, *Fluid Dynamics Research* **38**, 241–256, 2006.
2. Belytschko, T., Krongauz, Y., Organ, D., Fleming, M., Krysl, P., Meshless methods: an overview and recent developments. *Comput. Methods Appl. Mech. Eng*. **139**, 3–47. 1996.
3. Chikazawa, Y., Koshizuka, S. and Oka, Y., A particle method for elastic and visco-plastic structures and fluid-structure interactions, *Comput. Mech. J.*, **27**, 97–106, 2001.
4. Gotoh, H., Ikari, H., Memita, T. and Sakai, T., Lagrangian particle method for simulation of wave overtopping on a vertical seawall, *Coast. Eng. J.*, **Vol 47**, No 2âĂŞ3, 157–181, 2005.
5. Gotoh, H. and Sakai, T., Key issues in the particle method for computation of wave breaking, *Coastal Engineering J.*, **Vol 53**, No 2âĂŞ3, 171–179, 2006.
6. Heo, S., Koshizuka, S. and Oka, Y., Numerical analysis of boiling on high heat-flux and high subcooling condition using MPS-MAFL, *International Journal of Heat and Mass Transfer*, **Vol 45**, 2633–2642, 2002.
7. Koshizuka, S., Ikeda, H., and Oka, Y., Numerical analysis of fragmentation mechanisms in vapor explosions, *Nuclear Engineering and Design*, **Vol 189**, 423–433, 1999.
8. Koshizuka, S. and Oka, Y., Moving Particle Semi-implicit Method for Fragmentation of Incompressible Fluid, *Journal of Nuclear Science and Engineering* **123**, 421–434, 1996.
9. Koshizuka, S., and Oka, Y., Application of Moving Particle Semi-implicit Method to Nuclear Reactor Safety, *Comput. Fluid Dyn. J.*, **Vol 9**, 366–375, 2001.
10. Koshizuka, S., Oka, Y., and Nobe, A., Numerical Analysis of Breaking Waves Using the Moving Particle Semi-implicit Method, *International Journal for Numerical Methods in Fluids* **26**, 751–769, 1998.
11. Kim, K.S., Kim, M.H. and Park, J.C., Development of MPS (Moving Particle Simulation) method for Multi-liquid-layer Sloshing, *Journal of Mathematical Problems in Engineering*, http://dx.doi.org/10.1155/2014/350165. 2014.
12. Nomura, K., Koshizuoka, S., Oka, Y. and Obata, H., Numerical Analysis of Droplet Breakup Behavior Using Particle Method, *Journal of Nuclear Science and Technology* **38**, 1057–1064, 2001.
13. Shakibaeinia, A. and Jin, Y.C., A weakly compressible MPS method for simulation open-boundary free-surface flow. *Int. J. Numer. Methods Fluids*, **63**, 10, 1208–1232, 2009.
14. Shakibaeinia, A. and Jin, Y.C., A MPS Based Mesh-free Particle Method for Open Channel flow. *Journal of Hydraulic Engineering ASCE*. **137**, (11): 1375–1384. 2011.
15. Shakibaeinia, A. and Jin, Y.C., A mesh-free particle model for simulation of mobile-bed dam break. *Advances in Water Resources*, **34**, (6), 794–807, 2011.
16. Shakibaeinia, A. and Jin, Y. C., MPS Mesh-free Particle Method for Multiphase Flows, *Journal of Computational Methods Applied Mechanical Engineering* **13**, 229–232, 2012.
17. Shao, S., Lo, E.Y.M., Incompressible SPH method for simulating Newtonian and non-Newtonian flows with a free surface. *Adv. Water Resour*. **26**, 787–800. 2003.
18. Shibata, K. and Koshizuka, S., Numerical analysis of shipping water impact on a deck using a particle method, *Ocean Engineering J.*, **Vol 34**, 585–593, 2007.

19. Sueyoshi, M., Kashiwagi, M. and Naito, S., Numerical simulation of wave-induced nonlinear motions of a two-dimensional floating body by the moving particle semi-implicit method, *Journal of Marine Science and Technology*, **Vol 13**, 85–94, 2008.
20. Sun, Z., Xi, G., and Chen, X., A numerical study of stir mixing of liquids with particle method, *Chemical Engineering Science*, **Vol 64**, 341–350, 2009.
21. Sun, Z., Xi, G. and Chen, X., Mechanism study of deformation and mass transfer for binary droplet collisions with particle method, *Phys. Fluids*, **Vol 21**, 032106, 2009.
22. Tsubota, K., Wada, S., Kamada, H., Kitagawa, Y., Lima, R. and Yamaguchi, T., A Particle Method for Blood Flow Simulation âĂŞ Application to Flowing Red Blood Cells and PlateletsâĂŞ, *Journal of the Earth Simulator*, **Vol 5**, 2–7, 2006.

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 109-118.

Applying Fuzzy Self-Organizing Map and SPADE Algorithm in Indonesian Natural Disasters

Septianusa[1], M. Supriyaningsih[2], and F. Hakim[3]

[1]*Department of Statistics, Islamic University of Indonesia, Yogyakarta, Indonesia, Email : septianusa@gmail.com*
[2]*Department of Statistics, Islamic University of Indonesia, Yogyakarta, Indonesia, Email : maulinasupriyaningsih@gmail.com*
[3]*Department of Statistics, Islamic University of Indonesia, Yogyakarta, Indonesia, Email : hakimf@mipa.uii.ac.id*

Abstract. Indonesia is one of the most disaster prone countries in the world. The country faces multiple hazards such as earthquakes, tsunami, volcanic eruptions, floods, landslides, drought, and forest fires. The data of disaster occurrences stored in a database since 1815. The discovery of a useful knowledge from a large database is interesting. The aim of this paper is performing segmentation in regencies and cities in Indonesia based on the number of victims of the disasters, i.e. the victim was missing, death, injured, and the refugees using fuzzy self-organizing map. Then, the SPADE algorithm is used in each cluster as a result of fuzzy self-organizing map in order to find the sequential rules. These rules may help provides an overview of natural disasters in Indonesia to assist mitigation strategies to get better disaster management.

Keywords: Kohonen's Self-organizing Map**,** fuzzy self-organizing map, sequential rule mining, SPADE algorithm.

INTRODUCTION

Indonesia's geographical and geodynamics position has placed the country as one of the areas that are prone to natural disasters. Indonesia is located between the Pacific, Eurasian, and Australian tectonic plates. Geologically, Indonesia also located in a very unstable area known as "The Ring of Fire" (Figure 1.).

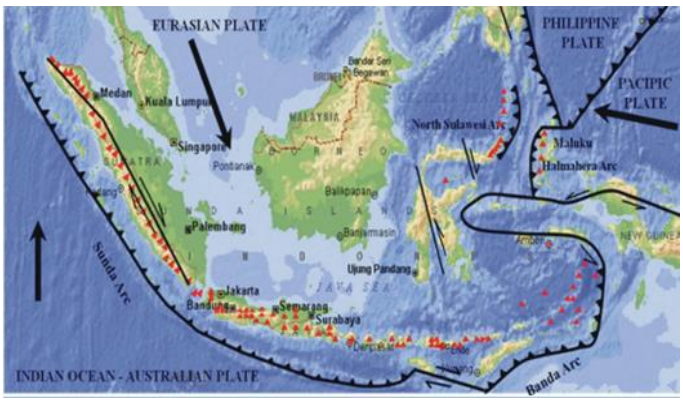


FIGURE 1. The ring of fire passed the islands in Indonesia

The Ring of Fire is a direct consequences of a phenomenon known as plate tectonics. When one of the plates moves, it causes earthquakes, volcanic eruptions and even tsunamis in Indonesia and the surrounding areas [19]. According to the report from Australian Agency for International Development (AusAID) and the Indonesia Agency for Disaster Management (BNPB), Indonesia is one of the most disaster prone countries in the world. The country faces multiple hazards such as earthquakes, tsunamis, volcanic eruptions, floods, landslides, droughts, and forest fires (3rd AIPA Caucus Report). In their reports, volcanic eruptions, earthquakes, tsunamis, floods and landslides kill thousands of people every year in Indonesia and leave more than 80,000 homeless. The threat of the disasters spread in various areas in Indonesia. Indonesia consists of 410 counties and 98 cities [26] with the different threat of disaster in each area and can occur at any time. The data of disaster occurrence stored in a database since 1815 [25]. With the availability of an increasingly abundant data of disasters which usually ignored by local government as a decision maker in their place of disaster happened, the discovery of a useful knowledge from a large database is ever more attracting attention. Therefore, the aim of this study is to perform segmentation in regencies and cities in Indonesia based on the number of victims of the disaster, i.e. the victim that was regarded as a missing, death, injured, and the refugees using the fuzzy self-organizing map. Moreover, the authors wanted to find the sequential patterns in the database of natural disasters that happened in Indonesia using the Sequential PAttern Discovery using Equivalence Classes (SPADE) algorithm. By finding patterns from databases using SPADE algorithm, the association of natural disasters experienced by each cluster or segments at a specific time will be appeared. These results are expected to help decision makers to solve unstructured or semi-structured disaster management problems.

RELATED WORKS

In [3] Kohonen self-organizing maps was showed as an algorithm that can classify samples of crude oils on the basis of gas chromatography–mass spectrometry (GC–MS) descriptors, in terms of geographical origin, with a high degree of accuracy.

In [21] the Self-organizing Kohonen's maps (SOMs) was introduced as a new approach to flood mapping using satellite synthetic-aperture radar (SAR) images and applied for SAR image segmentation and classification. The self-organizing map algorithm and its visualization capabilities was promoted to explore the value of 'hidden' information in such data sources and to gain insight into the complex behavior of the geography of a crime [7]. The approach is applied to a high-profile and unsolved murder series in the city of Jennings, Louisiana. Fuzzy set theory was combined with the unsupervised learning network model to create an unsupervised fuzzy self-organizing map (FSOM) model to take respective advantages of the learning function and the capability of handling uncertainty problems in human recognition processes [1]. An application temporal association rules to the multiple financial time-series was introduced [2]. To carry out a rules extraction stage, first, the process in this research involves finding significant events into multivariate time series considering time restrictions, and second, a search is made for sequences of episodes or items that are repeated amongst financial data.

In [11] the temporal association rules during the time intervals specified by user-given calendar schemas, has been discussed. The paper extends to the well-known Apriori algorithm, and also develops two optimization techniques to take advantage of the special properties of the calendar-based patterns. Their experiments showed that the algorithms and optimization techniques were effective.

In [20] the SPADE algorithm used to see the purchasing tendency of items by customer at a specific time period. SPADE algorithm also used to create a flood early warning system based on the data pattern of rain and flood. The final result is a rule that used as input to application early warning systems software to provide flood alert information [23]. In [10] a decision support framework built on rapid information collection and resource tracking functionalities was introduced. Their paper found an optimal solution for resource deployment and dispatching used operation research techniques based on information collected from emergency response service agencies.

In this study, the authors used fuzzy self-organizing map to perform segmentation in regencies and cities in Indonesia based on the number of victims of the disaster and the SPADE algorithm to find the sequential patterns in the districts that have been grouped by fuzzy self-organizing map.

PROBLEM FORMULATION

A. Kohonen's Self-organizing Map (SOM)

Kohonen's self-organizing map (SOM) network is an unsupervised learning neural network that maps an *n*-dimensional input data to a lower dimensional output map while maintaining the original topological relations [9], Figure 2.

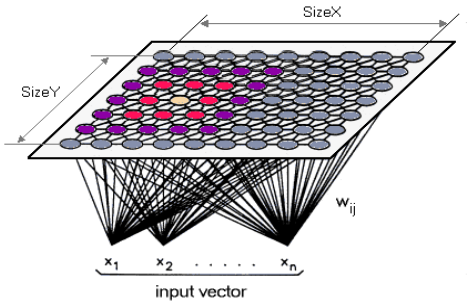


FIGURE 2. Structure of a rectangular SOM

The maps are trained in an unsupervised way using a competitive learning scheme. They are especially suitable for high-dimensional data visualization [18]. The Self-Organizing Map algorithm can be broken up into 6 steps [6]:

1. Each node's weights are initialized.
2. A vector is chosen at random from the set of training data and presented to the network.
3. Every node in the network is examined to calculate which ones weights are most like the input vector. The winning node is commonly known as the Best Matching Unit (BMU).

$$DistFromInput^2 = \sum_{i=0}^{i=n}(I_i - W_i)^2 \tag{1}$$

I = current input vector
W = node's weight vector
n = number of weights

4. The radius of the neighborhood of the BMU is calculated. The values start large. Typically it is set to be the radius of the network, diminishing each time-step.
   Radius of the neighborhood

$$\sigma(t) = \sigma_0 e^{(-t/\lambda)} \tag{2}$$

t = current iteration λ = time constant

$$\lambda = numIterations / mapRadius \tag{3}$$

σ_0 = radius of the map

5. Any nodes found within the radius of the BMU, calculated in 4), are adjusted to make them more like the input vector.
   New weight of a node

$$W(t+1) = W(t) + \Theta(t)L(t)(I(t) - W(t)) \tag{4}$$

   Learning rate

$$L(t) = L_0 e^{\left(-\frac{t}{\lambda}\right)} \tag{5}$$

6. The closer a node is to the BMU, the more its' weights are altered.
   Distance from BMU

$$\Theta(t) = e^{\left(-distFromBMU^2/\left(2\sigma^2(t)\right)\right)} \tag{6}$$

7. Repeat step 2 for N iterations until the feature map stops changing.

B. Fuzzy SOM

The procedure of the application of fuzzy set theory in the context of SOM for mining and visualizing uncertainties is as follows [8]:

1. Fuzzy competition: in fuzzy competition, input vector can belong to more than one weight, and associated with each element by a set of membership levels. The standard function is:

$$U_x = \frac{1}{\sum_j \left(\frac{d(weight_k, x)}{d(weight_j, x)} \right)^{\frac{2}{m-1}}} \tag{7}$$

   Where U_x is the membership value of each input vector x to all weights, $j = 1,2,...,w$, and m is the level of cluster fuzziness. In fuzzy competition, all the neurons are winning neurons with the membership degree ranging [0..1].

2. Fuzzy cooperation: in fuzzy cooperation, all winning neurons cooperate with their neighboring neurons in terms of the membership degree as follows:

$$h(j,i) = U_{xi} \times \exp\left(\frac{-d_{j,i}^{\ 2}}{2\sigma^2}\right) \quad i,j = 1,2,...,n; i \neq j \tag{8}$$

where j is the number of the cooperating neighbor neurons, i is the number of the winning neurons including all the neurons with different membership degrees, $h(j,i)$ is the topological area centered around the winning neuron i and the cooperating neuron j, U_{xi} is the membership value of input vector x from i^{th} winning neuron, σ is the size of the neighborhood that needs to decrease with time.

3. Fuzzy adaption: the fuzzy adaption is the weight update by:

$$w_j = w_j + U_j \times \left(\eta h(j,i) \times (x - w_j)\right) \quad i,j = 1,2,...,n; i \neq j \tag{9}$$

Where U_j is the membership value of input x from neuron j.

These three steps are repeated until the maximum number of iterations is reached smaller than a predefined threshold.

C. Sequential Rule Mining

Sequential Pattern Mining finds interesting sequential patterns among the large database and finds out frequent subsequences as patterns from a sequence database [17]. Sequential rule mining or temporal rule mining is an alternative of sequential pattern mining that addresses the problem of prediction. A sequential rule (also called episode rule, temporal rule or prediction rule) indicates that if some event(s) occurred, some other event(s) are likely to occur with a given confidence or probability [4]. Using sequential rules, we can know the series of events that will usually occur after a series of previous ones [16].

Definition 1 Let min_s and min_c represent minimum support threshold and minimum confidence threshold respectively, if and only if during [t_s, t_e], support ≥ min_s, confidence ≥ min_c, rule X→Y is a temporal association rule, which could be described as X→Y (support, confidence, [t_s, t_e]) [12].

No matter what kinds of tools or algorithms you select, strong temporal association rule mining can be divided into 4 steps [12].

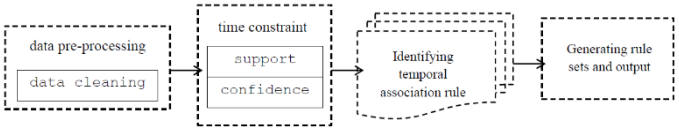


FIGURE 3. Temporal association rule mining steps

1. Data pre-processing. It includes data cleaning, data integration, data exchange and data reduction. In virtue of data pre-processing, we can get high quality data mining object.
2. To find the frequent itemsets which have the support no less than min_s.
3. To generate association rules with frequent itemsets. It is different to generate association rules without time, because it adds time information on frequent itemsets. So here the association rules are temporal ones.
4. To generate rule sets and output.

D. SPADE Algorithm

SPADE (Sequential PAttern Discovery using Equivalence Classes) is an apriori-based, vertical formatting method which also is another approach for mining frequent sequential patterns. In this approach, the sequence database is transformed into a vertical id-list database format, in which each item is associated with a list of all sequence_id and transaction_id. The SPADE algorithms are as follows [13]:

Step 1: Make the first pass over the sequence dataset D to yield all the 1-element frequent sequences

Step 2: Repeat until no new frequent sequences are found

Candidate Generation: Merge pairs of frequent frames subsequences found in the (k-1)th pass to generate candidate sequences that contain k items

Candidate Pruning: Prune candidate k-sequences that contain infrequent (k-1)- subsequences

Support Counting: Make a new pass over the sequence dataset D to find the support for these candidate sequences of the frames

Candidate Elimination: Eliminate candidate k-sequences ($k = \sum_j |\alpha_j|$, a sequence with k items) whose actual support is less than minseq

To extract the sequential patterns, SPADE needs three database scans: the first scan aims at finding the frequent items, the second at finding the frequent sequences of length two and the last one associate to frequent

sequences of length two, a table of the corresponding sequence_id and itemset_id (or transaction_id) in the database (called id-lists) [5].

E. Knowledge and Discovery Process

This study aims to get valuable information from Data pile of natural disasters in the database. There are several phase in discovery process, first, is a preprocessing phase. Variables selection and cleaning data that will be used is done at this phase. Then the data is transformed and made to the same scale for each variable (fuzzification process), with the objective of this process that there are no dominant variable.

At the data mining procedure, when all variables have same measurement and ranges, then the association rules extraction process is done. Fuzzy SOM formed clusters of disaster area based on the characteristics of each region. Then for recognizing natural disasters pattern sequences use SPADE algorithm. The knowledge discovery can be seen in Figure 4.

After all clusters and associated rules are formed, the user can select and evaluate rules that are considered useless or useful (keeping the most valuable rules into the database). According to the characteristics of each region and the potential association relationships, authorities can use it as a basis of decision making. With all the events that may be occurred, these rules could be developed into better strategies and manage a disaster with a variety of purposes.

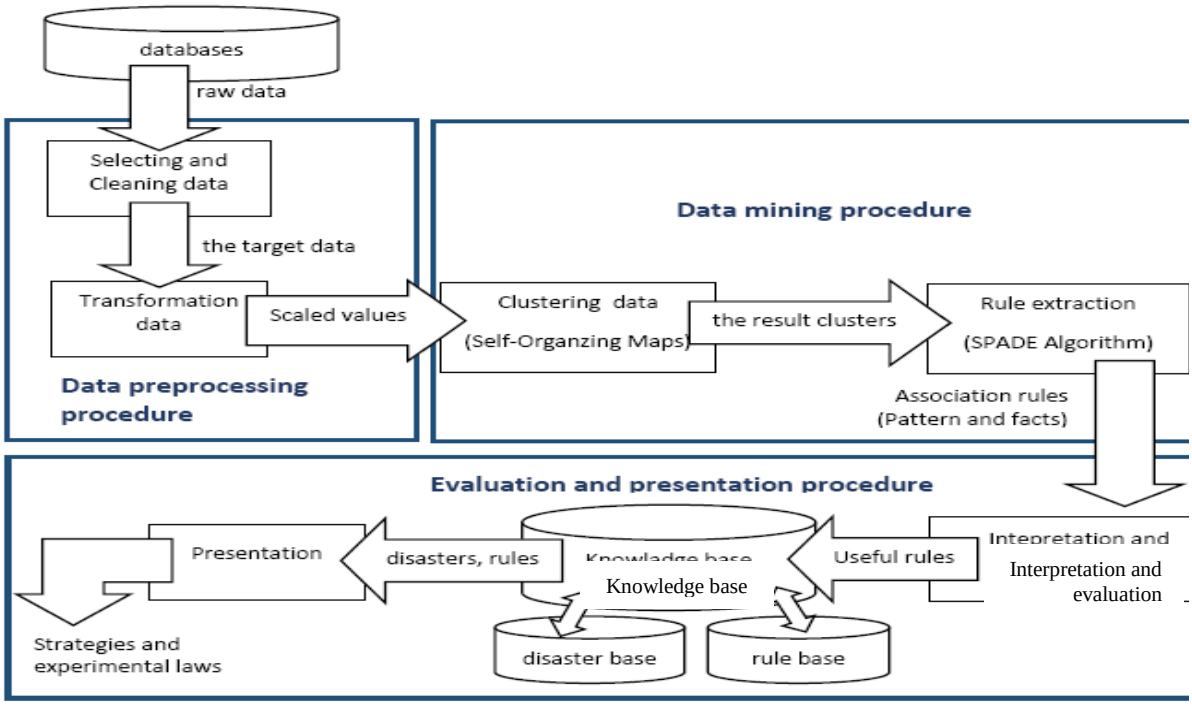


FIGURE 4. Knowledge Discovery

NATURAL DISASTERS OVERVIEW IN INDONESIA

In this paper the authors used data from 1908 to 2015 of 568 districts in Indonesia. The data are taken directly from Indonesian natural disaster database provided from DIBI [24]. Variables used in this study is the district, type of disaster, the date of the disaster, victims died, victims lost, injured, and refugees (displaced victims). Data were already taken and processed as Figure 4. The overview of natural disasters in Indonesia can be described as follows:

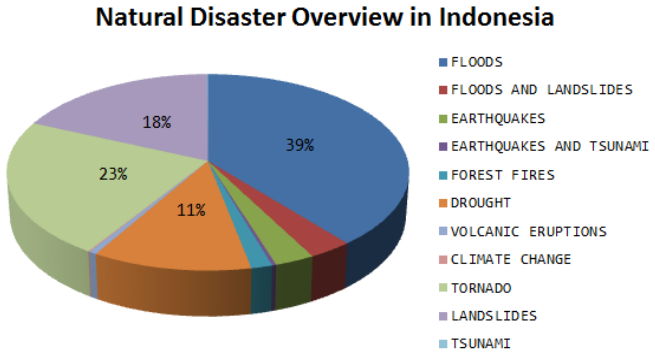


FIGURE 5. Pie chart of natural disaster overview

Based on Figure 5, the most frequent natural disaster that occurred in Indonesia is floods (39%). Then, the tornado (23%), landslides (18%) and drought (11%).

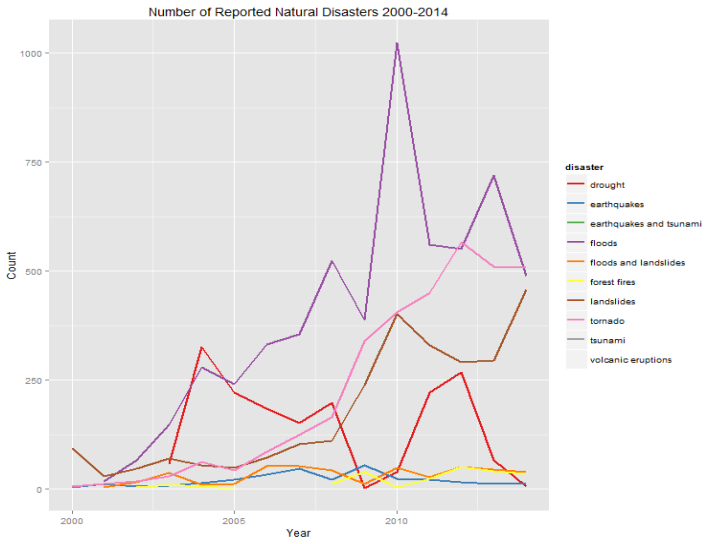
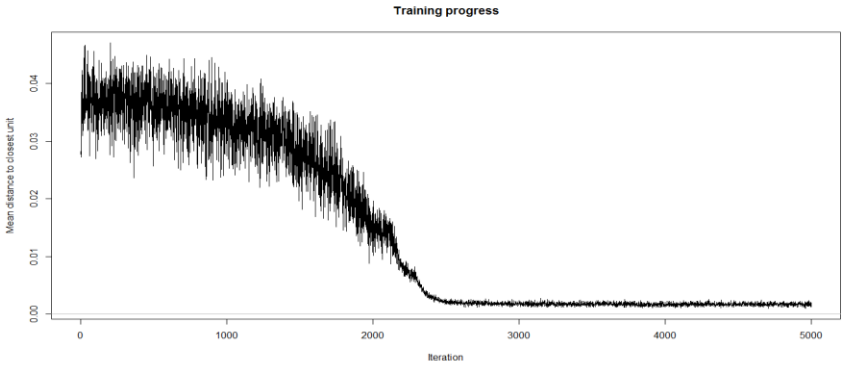


FIGURE 6. Number of reported natural disasters

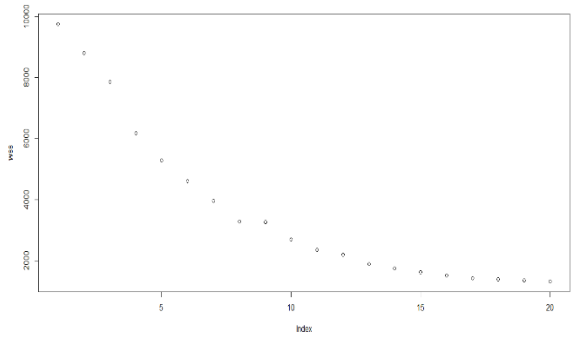
Based on Figure 6, in the period of 2010 to 2014, floods occured most frequently in 2010. In 2014, the frequency of occurrence of floods has less decreased than in 2013. As well as the frequency of floods, drought has also decreased in 2014. A disaster which tend to increase during this period, is landslides.

F. Districs Clustering

Data related to victims and the frequency of occurrences in each natural disaster is used as variables (columns) in the clustering phase. Each variable is formed in a new scale using fuzzy rules functions, victims died, victims lost, injured, displaced victims, and the frequency of occurrence was made into a new form for each type of disaster. This phase produces a type of disaster in each district with its weight of total occurrences and calls this result as nine new impact variables. These nine new variables namely the impact of landslides (TL), the impact of flooding (BA), the impact of floods and landslides (BATL), the impact of the earthquake (GB), the impact of the tsunami (TS), the impact of the earthquake and tsunami (GBTS), the impact of volcanoes (GA), the impact of forest fires (KB), and the impact of the tornado (PB).



(a)



(b)

FIGURE 7. Figure (a) on the upper is a training progress. Figure (b) on the lower shows the number of cluster

Clustering process is done using fuzzy self-organizing maps based on characteristics of disasters in each districts. From 5000 iterations have been derived average distance unit cluster is 0.0187 (Figure 7. (a)), with the number of clusters are formed is 17 clusters (Figure 7. (b)).

G. Sequential Pattern Mining

Sequential pattern mining has been used to discover patterns the sequences of natural disasters in Indonesia. Districts that have been grouped are analyzed using SPADE algorithm. In the SPADE algorithm, a sequence ID is the result of all districts that have been grouped into 17 clusters which was formed by Fuzzy SOM. Subsequently, the districts regarded as the event ID and SIZE is assumed from the accumulation of the sequence of natural disasters in each district. The pattern sequence of natural disasters in Indonesia which was performed using arulesSequences [27] and Kohonen packages [28] of R software [29] can be shown in the form of rule events indicated by the support value (Table 1.).

TABLE 1. EVENT SEQUENCE

No	Support >0.8	
	Sequence	**Support**
1	<{Floods}>	1
2	<{Floods and Landslides}>	0.875
3	<{Fires}>	0.875
4	<{Tornado}>	0.9375
5	<{Landslides}>	0.9375
6	<{Lanslides, Floods}>	0.9375
7	<{Landslides, Floods and Landslides}>	0.875
8	<{Tornado, Landslides}>	0.875

No	Support >0.8	
	Sequence	Support
9	<{Tornado, Landslides, Floods}>	0.875
10	<{Landslides, Floods, Floods and Landslides}>	0.875
11	<{Tornado, Floods}>	0.9375
12	<{Floods, Fires}>	0.875
13	<{Floods, Floods and Landslides}>	0.875

Frequent sequences that have only one item indicate that the item is an item which is most often occurred. For example, <{Floods}>, <{Floods and Landslides}>, <{Fires}>, <{Tornado}>, and <{Landslides}>. A sequential pattern is said often occurred if the sequential patterns have high support values. The most frequent item which is occurred in Indonesia is "Floods" with *support* 100%.

Analyzing data of DIBI database using a combination of FSOM and SPADE are producing thousands of rules. Some examples of the rules are:

```
Rule                          Supp    Conf    lift
<{BA}> => <{KR}>              0.5     0.5     0.67
<{KR}> => <{KR}>              0.5     0.67    0.89
<{PB}> => <{KR}>              0.5     0.53    0.71
<{TL},{PB}> => <{KR}>         0.375   0.75    1.0
<{KT,BA},{GB}> => <{PB,GB}>   0.3125  1       1.45
<{KT},{PB,KB}> => <{GB}>      0.3125  1       1.33
```

A strong rule is the rule that have lift ratio value greater than 1.The greatest lift ratio value can be found in the fifth rule. For example, `<{PB}> => <{KR}>` with *confidence* 53% meaning that *if the tornado occurred, then the possibility of drought also occurred is 53%*.

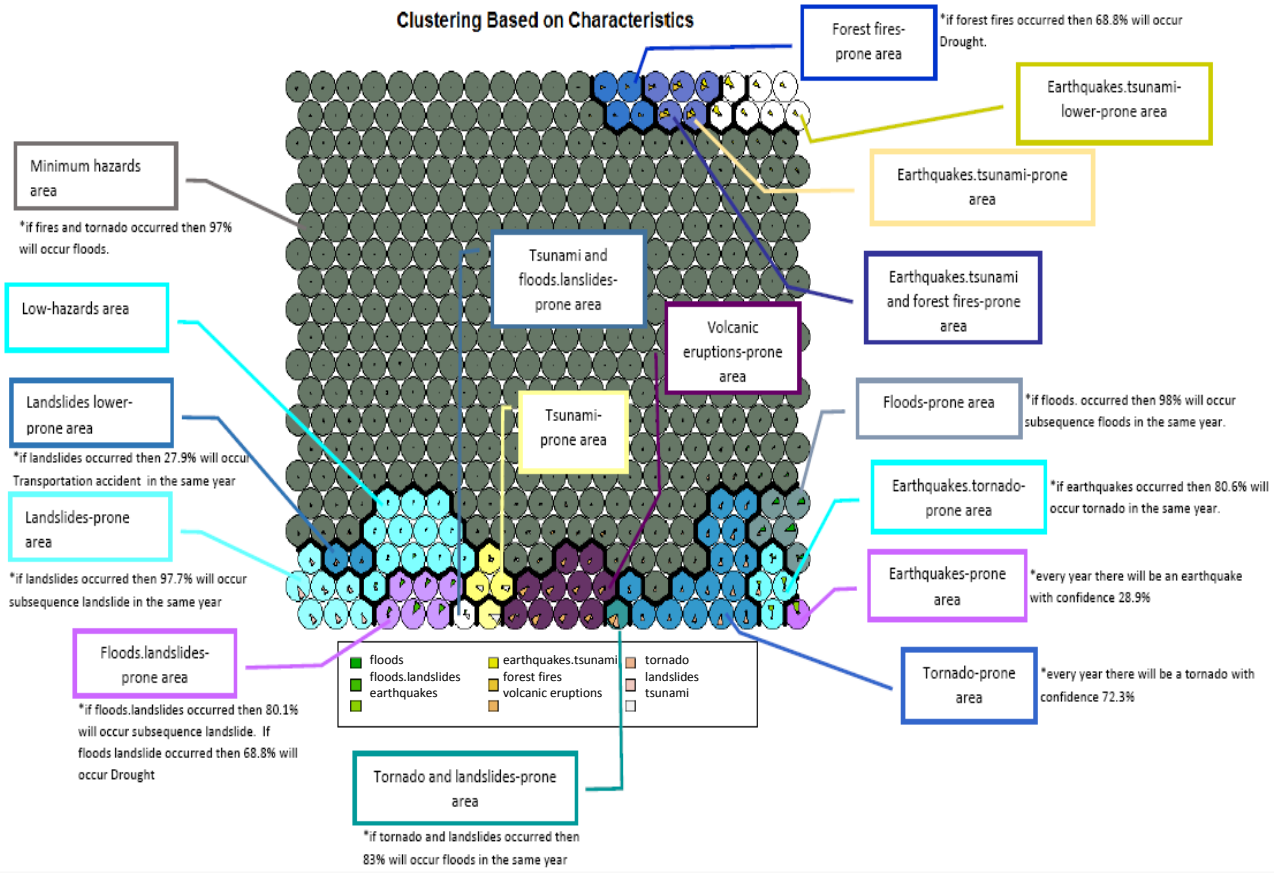


FIGURE 8. Map of Clustering Based on Characteristics

The results of the cluster should be combined with the pattern of sequence of events to find an magnificent answer to the problems of natural disaster management in Indonesia. Hence, the displayed results will simplify decision making of government policy. Some examples of this combination (Figure 8.) are: 1) purple cluster, is prone to floods-landslides. At this cluster has been found the rules, if floods-landslides occurred, then subsequent landslides would occur with possibility 80.1%. In addition, also known if floods-landslides happened, then drought also happened with possibility 68.8%. 2) Blue cluster, is prone to tornadoes. Every year the districts in this cluster will encounter tornado disaster with possibility 72.3%.

CONCLUSION

This study provides an overview and pattern of natural disasters in Indonesia. Clustering based on FSOM divided the districts into clusters, and the sequential rule to discover the pattern of disasters and explain the relationship between each occurrence of disasters. Furthermore, authorities recognize the characteristics of each district disasters in order to decrease the loss of impact of disaster. These rules may help provides an overview of natural disasters in Indonesia to assist mitigation strategies to get better disaster management. Further research will be aimed at adding or replacing the variables used in the fuzzification process.

REFERENCES

1. S.C. Chi, R.J. Kuo, and P.W. Teng, A fuzzy self-organizing map neural network for market segmentation of credit card, *Proceedings of the IEEE International Conference on Systems, Man and Cybernetics*, vol. 5, 2000, pp. 3617-3622.
2. C. Dante, F. J. Martinez-de-Pison, and P. Alpha, Finding temporal associative rules in financial time-series: A case of study in Madrid Stock Exchange (IGBM), *Proceedings of the 9th WSEAS Int. Conf. on Computational Intelligence, Man-Machine Systems and Cybernatics (CIMMACS '10)*, 2010, pp. 60-68.
3. A. M. Fonseca, J. L. Biscaya, J. Aires-de-Sousa, and A. M. Lobo, Geographical classification of crude oils by Kohonen self-organizing maps," *Analytica Chimica Acta*, vol. 556, 2006, pp. 374-82.
4. P. Fournier-Viger, U. Faghihi, R. Nkambou, and E. M. Nguifo, CMRules: Mining Sequential Rules Common to Several Sequences, *Knowledge-based Systems*, vol. 25, 2012, pp. 63-76.
5. K. Gouda and M. Hassaan, Mining Sequential Patterns in Dense Databases, *International Journal of Database Management Systems*, vol. 3, no. 1, 2011, pp. 179–194.
6. S. M. Guthikonda, *Kohonen Self-Organizing Maps*, Wittenberg University, 2005.
7. J. Hagenauer, M. Helbich, M. Leitner, J. Ratcliffe, S. Chainey, and R. Edwards, Data Mining of Collaboratively Collected Geographic Crime Information Using an Unsupervised Neural Network Approach, *AutoCarto 2012*, 2012.
8. A. Karami and M. Guerrero-Zapata, Mining and Visualizing Uncertain Data Objects and Named Data Networking Traffics by Fuzzy Self-Organizing Map, *Proceedings of the Second International Workshop on Artificial Intelligence and Cognition (AIC 2014) International Workshop on Artificial Intelligence and Cognition*, 2014, pp. 156-163.
9. M. Y. Kiang, M. Y. Hu, and D. M. Fisher, An extended self-organizing map network for market segmentation-a telecomunication example, *Decision Support Systems*, vol. 42, 2006, pp. 36–47.
10. R. Kondaveti and A. Ganz, Decision support system for resource allocation in disaster management, *Proceedings of the 31st Annual International Conference of the IEEE EMBS*, 2009, pp. 3425-3428.
11. Y. Li, P. Ning, X. Sean-Wang, and S. Jajodia, Discovering calendar-based Temporal Association Rules, *Data and Knowledge Engineering Journal (Elsevier)*, vol. 44, no. 2, 2003, pp. 193–218.
12. Z. Liang, T. Xinming, L. Lin, and J. Wenliang, Temporal Association Rule Mining Based On T-Apriori Algorithm and Its Typical Application, *Proceedings of International Symposium on Spatio-temporal Modeling, Spatial Reasoning, Analysis, Data Mining and Data Fusion*, vol. 5, 2005, pp. 1-6.
13. H. Mathur, Sequential Mining of Multimedia Images by Using SPADE Algorithm, *International Journal of Computer Science and Information Technologies*, vol. 4, 2013, pp.791-95.
14. M. Parikh, B. Chaudhari, and C. Chand, A Comparative Study of Sequential Pattern Mining Algorithms, *Internatioanl Journal of Application or Innovation in Engineering & Management (IJAIEM)*, vol. 2, no. 2, 2013, pp. 103-109.
15. K. S. Patil and S. S. Patil, Sequential Pattern Mining Using Apriori Algorithm & Frequent Pattern Tree Algorithm, *IOSR Journal of Engineering (IOSRJEN)*, vol. 3, no. 1, 2013, pp. 26–30.
16. T. T. Pham, J. Luo, T. P. Hong, and B. Vo, An Efficient Algorithm For Mining Sequential Rules With Interestingness Measures, *International Journal of Innovative Computing, Information and Control*, vol. 9, 2013, pp. 4811-4824.
17. A. Reshamwala and S. Mahajan, Improving Efficiency of Apriori Algorithms for Sequential Pattern Mining, *Bonfring International Journal of Data Mining*, vol. 4, 2014, pp. 1-6.
18. F. P. Romero, A. Peralta, A. Soto, J. A. Olivas, and J. Serrano-Guerrero, Fuzzy optimized self-organizing maps and their application to document clustering, *Soft Computing*, vol. 14, no. 8, 2010, pp. 857-867.
19. H. Saul, Mount Kelud eruption: Why is Indonesia the hottest spot on the Ring of Fire?, *Independent*, viewed 12 February 2015, 2014, http://www.independent.co.uk/news/world/asia/mount-kelud-eruption-why-is-indonesia-the-hottest-spot-on-the-ring-of-fire-9128148.html.

20. R. Sijabat, *Penentuan Pola Sekuensial DataTransaksi Pembelian Menggunakan Algoritme Spade*, Bogor Agricultural University, Indonesia, 2011.
21. S. Skakun, A Neural Network Approach To Flood Mapping Using Satellite Imagery, *Computing and Informatics*, vol. 29, 2010, pp. 1013-1024.
22. T. Slimani and A. Lazzez, Sequential Mining: Patterns and Algorithms Analysis, *International Journal of Computer and Electronics Research*, vol. 2, Issue 5, 2013, pp. 639-647.
23. W. Suwarningsih and E. Suryawati, Pembangkitan Pola Data Cuaca Untuk Sistem Peringatan Dini Banjir, *Informatika, Sistem Kendali dan Komputer (INKOM)*, vol. 6, 2012, pp. 9-13.
24. Badan Nasional Penanggulangan Bencana, *Data dan Informasi Bencana Indonesia*, viewed 13 January 2015, http://dibi.bnpb.go.id/data-bencana.
25. Badan Nasional Penanggulangan Bencana, *Data dan Informasi Bencana Indonesia*, viewed 13 January 2015, http://dibi.bnpb.go.id/.
26. Komite Pemantauan Pelaksanaan Otonomi Daerah, *Daftar Provinsi, Kabupaten & Kota di Indonesia (Mei 2013)*, Viewed 13 January 2015, http://www.kppod.org/datapdf/daerah/daerah-indonesia-2013.pdf.
27. C. Buchta and M. Hahsler, *arulesSequences: Mining frequent sequences*, viewed 13 January 2015, http://cran.r-project.org/web/packages/arulesSequences/arulesSequences.pdf.
28. R. Wehrens, *Kohonen: Supervised and unsupervised self-organising maps*, viewed 13 January 2015, http://cran.r-project.org/web/packages/kohonen/kohonen.pdf.
29. R-3.0.2 for Windows (32/64 bit), viewed 13 January 2015, http://cran.r-project.org/bin/windows/base/old/3.0.2/.

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 119-125.

On the Mathematical Modeling of Splicing Systems in DNA Computing: A Review

Nor Haniza Sarmin[1], Muhammad Azrin Ahmad[2] and Wan Heng Fong[3]

[1]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia, Email: nhs@utm.my*
[2]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia, Email: azrinahmad1@gmail.com*
[3]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia, Email: fwh@utm.my*

Abstract. A splicing system involves the process of cutting and pasting on DNA molecules with the presence of restriction enzymes and ligase, respectively. A mathematical model of the splicing system has been developed by using the concept of formal language theory, which is a branch of theoretical computer science and applied discrete mathematics, and informational macromolecules. The splicing system consists of a set of alphabets, a set of rules and a set of initial strings. Over the years, the model of splicing system has grown intensively. In this research, there are two issues that are of interest of the researchers - either a model based on the generation of language, or a model based on the preservation of biological traits of the splicing process. In this paper, basic concepts on different types of splicing systems are presented. The current trend and results on the splicing system with other related fields such as computer science and molecular biology are given and discussed.

Keywords: mathematical modeling, splicing systems, DNA computing.

INTRODUCTION

Biomolecular computing, also known as DNA computing, has replaced traditional silicon-based computer technologies by using DNA, biochemistry and molecular biology hardware. DNA computing has implemented two fundamental features of DNA which are Watson-Crick complementarity and the massive parallelism of DNA strands. The potential of handling the massive and complex computations made the DNA computing promising tool and devices of computing. Adleman [1] has initiated the study on DNA computing to solve the seven point Hamiltonian path problems. Splicing system and sticker system have been recognised as languages generating devices in formal language theory for DNA-based computing devices. They are modeled mathematically to generate some strings over the four-symbol alphabets similar to the deoxyribonucleotide pairs [1].

In addition, formal language theory, which is originally developed from applied discrete mathematics and theoretical computer science, is an abstraction of the general characteristics of programming languages. A formal language is a set of strings over a finite alphabet which follows some rules in order for those strings to form a language. The study of families of formal languages leads to the formal language theory [1]. Surprisingly, the cutting and pasting phenomenon on DNA molecules can be represented into some terms of the formal language that inspired the study on the mathematical modelling of the splicing system. Head, who is the pioneer on this study, had established the interdisciplinary study between formal language theory and the study of informational macromolecules in 1987 [2]. As years passed by, this study has developed tremendously. The focus of the researchers has been split into two directions: on the generation of languages or the preservation of biological traits in the splicing process [3].

In this paper, the fundamental knowledge of formal language theory and some related molecular biological terms are presented. A mutual relation that exists in these fields is then explained. Besides that, the historical review on the different models and some types of splicing systems, various types of splicing languages and more are discussed.

PRELIMINARIES

A mathematical model of the splicing system which has been initiated by Head [2] comprises of four different set of elements such as a set of alphabets, a set of initial strings, and two sets of rules which are presented in triple forms. Therefore, some fundamental terms regarding formal language theory are given.

In the following, some definitions related to formal language theory which will be used throughout this paper, namely alphabet, string and language, are listed below.

Definition 1. [1] *An alphabet, A, is a finite, nonempty set of symbols.*

Definition 2. [1] *A string is a finite sequence of symbols from the alphabet.*

Definition 3. [1] *A set of strings all of which are chosen from some A^*, where A is a particular alphabet, is called a language and A^* denotes the set of all strings over an alphabet A, which is obtained by concatenating zero or more symbols from A.*

The operations involved between the splicing languages are called concatenations. The concatenation between two languages, L_1 and L_2 has been given in [4] where $L_1L_2 = \{xy \mid x \in L_1, y \in L_2\}$. The operation between the languages are written in the following general form:

$$L^0 = \varnothing,$$
$$L^{i+1} = LL^i,\ i \geq 0,$$
$$L^* = \bigcup_{i=0}^{\infty} L^i \left(\text{the * - Kleene closure}\right),$$
$$L^+ = \bigcup_{i=1}^{\infty} L^i \left(\text{the + - Kleene closure}\right).$$

Next, some molecular biological terms are introduced and the relations with formal language theory are then emphasized.

The presence of DNA in each living organism has offered variety in them such as their physical looks. It is important to study the structure of DNA as knowing the structure leads to the discovery of its important functions and also its significant role in the hereditary processes [5]. This unique molecule plays two important roles which are coding for the production of proteins and also self-replication that transfers information from the parent cells to the offspring cells [4]. This is to ensure some similar traits are inherited to the offsprings from the parent. Basically, a DNA molecule is composed of nucleotides. A complete nucleotide is made up of three distinct components which are a phosphate group, a sugar group (deoxyribose) and a nitrogenous base. There are four types of nitrogenous bases which can be grouped as pyrimidine: Cytosine (C) and Thymine (T) and purine: Adenine (A) and Guanine (G) [6]. By Watson-Crick complementarity, the only possible pairings allowed between the nucleotides are A with T, C with G and vice versa. In 1940s, Erwin Chargaff had performed the base compositions of DNA from various species and concluded the amount of A and T was approximately equal and likewise for C and G [5] which reason the pairings. In reality, the representation of a double-stranded DNA (dsDNA) molecule as two linear strands bound together by Watson-Crick complimentarity is already a simplification since in a DNA molecule, the two strands wind around each other to form the famous double helix [4].

Besides that, restriction enzymes are enzymes isolated from the bacteria which have a significant role on DNA molecules [6]. The restriction enzyme cuts the DNA molecules within a particular nucleotide sequence called restriction sites [5]. DNA molecules also consist of hundreds of thousands of nucleotides. Besides cutting the DNA molecules, restriction enzymes also help recognise specified region of the entire DNA molecules and locate the restriction site. When the restriction enzyme acts on the DNA molecule that has a cutting site, two fragments with complementary sticky ends or blunt ends will exist [2]. The pasting process then takes place with the presence of ligase when two conditions are met: the two fragments end with complementary bases, and the two fragments are from the same overhang. Then, either a new hybrid DNA or the same DNA molecule is formed. After a cutting process has taken place, there will be three types of cuts namely $5'$- overhang, blunt end or $3'$- overhang. They can be categorised in left ($5'$- overhang and blunt end) or right ($3'$- overhang) pattern [2].

Mathematically, the complementary bases which are known as *a*, *c*, *g* and *t* are presented as a set of alphabets, *A*. Besides, the dsDNA molecules are represented as a set of initial strings, *I* and restriction enzymes which act upon dsDNA molecules are represented as a set of rules: set *B* represents $5'$- overhang and blunt end cut while set *C* represents $3'$- overhang cut. Meanwhile, dsDNA molecules obtained from the splicing process are represented as a language. In the next section, the definitions of the splicing system by Head, Paun, Pixton, Goode-Pixton and Yusof-Goode (Y-G) are presented. The differences among the models are then discussed.

DEVELOPMENT OF SPLICING SYSTEMS

In this section, the pioneer model proposed by Head is first given. Besides, the differences with the other models are discussed. In addition, two examples are proposed where the first is inspired by molecular biology. The second example establishes the relation between molecular biology and formal language theory.

The inspiration of the mathematical modelling of splicing system is triggered by the presentation of DNA molecules as a series of alphabets adenine (A), cytosine (C), guanine (G) and thymine (T). Hence, by considering the actual process of DNA splicing, Head carefully designed the following mathematical model of the splicing system.

Definition 4. [2] *A splicing system S = (A, I, B, C) consists of a finite alphabet A, a finite set I of initial strings in A*, and finite sets B and C of triples (c, x, d) with c, x and d in A*. Each such triple in B or C is called a pattern. For each such triple the string cxd is called a site and the string x is called a crossing. Patterns in B are called left patterns and patterns in C are called right patterns. The language L = L(S) generated by S consists of the strings in I and all strings that can be obtained by adjoining the words ucxfq and pexdv to L whenever ucxdv and pexfq are in L and (c, x, d) and (e, x, f) are patterns of the same hand. A language L is a splicing language if there exists a splicing system S for which L = L(S).*

As years passed by, researchers made intensive works on the concept of splicing system introduced by Head. Several model were developed namely Paun [7], Pixton [8], Goode-Pixton [9] and Y-G [10] splicing systems. Initially, Head splicing system was bounded to finite case only. Paun had introduced a new formalism where it is found to be a powerful approach at some cases due to its rule that is set to consider also the infinite cases. Comparison between Paun's and Head's model is then made in [11]. It has been shown that Paun's model works terrific in theory but not biologically. This difference will be further discussed in the next section. The definition of Paun's splicing system is presented as follows.

Definition 5. [7] *A Paun splicing scheme is defined as a pair* $\sigma(A,R)$ *where A is an alphabet and* $R \subseteq A^* \# A^* \$ A^* \# A^*$ *is a set of splicing rules, for #, $ two special symbols not in A. Two initial strings x and y in* A^* *can be spliced via rule r in R to produce the following language:*

$$\sigma(x,y) = \{uu_1u_4v' \mid x = uu_1u_2v,\ y = u'u_3u_4v' \text{ for some } u,u',v,v' \in A^* \text{ and } u_1 \#u_2 \$v_1 \#v_2\}.$$

Pixton splicing system [8] had been introduced in the same year as Paun's, a substitution approach is used where an intervening factor called β is applied in between the left part of the first string and the right part of the second string. The following is the definition of Pixton splicing system.

Definition 6. [8] *A Pixton splicing system is defined as a pair* $\zeta = (R, L_0)$ *where R is a splicing scheme (A, r) and* L_0 *is a set of strings, called initial language generated from* A^**. The rule r in R is presented in triple of* $(\alpha, \alpha'; \beta)$*. Given two strings* $\varpi = \xi a \eta$ *and* $\varpi' = \xi' a' \eta'$*, then by applying the rule r to these strings, the string* $\xi \beta \eta'$ *is produced.*

Bonizzoni *et al.* [11] later claimed that Pixton's concept was a mere substitution operation regarding splicing since it is against the earliest concept proposed by Head and Paun. To overcome the misunderstanding of the previous concept, Goode and Pixton came out with Goode-Pixton splicing system. The following is the definition of Goode-Pixton splicing system.

Definition 7. [9] *Let A denote a finite alphabet. A splicing rule, also called a rule for simplicity, is a quadruple* $(u,v:u',v')$ *where u, v,* u' *and* v' *come from* A^*. *Given a rule r =* $(u,v:u',v')$ *and two strings w = xuvy and* $w' = x'u'v'y'$ *in* A^*, *w and* w' *can be spliced via r, generating the splicing product* $xuv'y'$. *This is denoted by the following shorthand:*

$$\{xuvy, x'u'v'y'\} \xrightarrow{(u,v;u',v')} xuv'y'.$$

A concern has arisen since it is ambiguous to determine whether the rule lies in the left or right pattern. Therefore, a new notation for writing rules in a splicing system, and a new extension of splicing system rooted from Head's and Goode-Pixton's version of splicing system which is called the Yusof-Goode (Y-G) splicing system is introduced. The following is the definition of Y-G splicing system.

Definition 8. [10] *A splicing system* $S = (A, I, R)$ *consists of a set of alphabets A, a set of initial strings I in* A^* *and a set of rules,* $r \in R$ *where r = (u, x, v: y, x, z). For* $s_1 = \alpha uxv\beta$ *and* $s_2 = \gamma yxz\delta$ *elements of I, splicing* s_1 and s_2 *using r produces the initial string I together with* $\alpha uxz\delta$ and $\gamma yxv\beta$, *presented in either order where* $\alpha, \beta, \gamma, \delta, u, x, v, y$ and $z \in A^*$ *are the free monoids generated by A with the concatenation operation and 1 as the identity element.*

Here, *R* represents a set of rules in either left pattern (*u*; *x*, *v*: *y*; *x*, *z*), right pattern (*u*, *x*; *v*: *y*, *x*; *z*) or both (*u*, *x*, *v*: *y*, *x*, *z*). This version of splicing system presents the transparent behaviour of the DNA biological process. It is claimed so because the splicing model itself is inspired by the characteristics of the restriction enzymes contained in the model.

Next, to portray the relation of splicing system and molecular biology, a molecular based example that takes place on dsDNA molecules with the presence of restriction enzyme, ligase and other appropriate substances is given.

Example 1. Let the dsDNA molecules be presented as:

$$\begin{array}{l} 5'-\text{MMMM}CCGC\text{MMMM}-3' \\ 3'-\text{WWWW}GGCG\text{WWWW}-5 \end{array}$$

where the symbol M denotes any arbitrary symbols from a set of alphabets that fulfilled the Watson-Crick complementarity. When a restriction enzyme namely *Aci*I is added to the solution containing the molecule, the dsDNA molecule is spliced as follows:

$$\begin{array}{l} 5'-\text{MMMM}C \quad CGC\text{MMMM}-3' \\ 3'-\text{WWWW}GGC \quad G\text{WWWW}-5 \end{array}.$$

Therefore, the following two new molecules are formed with the presence of a ligase despite the initial dsDNA molecule:

$$\begin{array}{l} 5'-\text{MMMM}CCGG\text{WWWW}-3' \\ 3'-\text{WWWW}GGCC\text{MMMM}-5 \end{array} \text{ and } \begin{array}{l} 5'-\text{WWWW}CCGG\text{MMMM}-3' \\ 3'-\text{MMMM}GGCC\text{WWWW}-5 \end{array}.$$

The molecular example above can be expressed in term of Head splicing system as shown in Example 2.

Example 2.
Let $S = (A, I, B, C)$ be a splicing system where $A = \{a, c, g, t\}$, $I = \{mmmmccgcmmmm\}$, $B = \{(c, cg, c)\}$ and $C = \phi$. When splicing occurs, the following strings are formed:

$$\{mmmmccgcmmmm\} \xrightarrow{B,C} I \cup \{mmmmccggwwww, wwwwgcgcmmmm\}.$$

The splicing language, $L(S)$ generated from this splicing language is given by

$$L(S) = I \cup \{mmmmccggwwww, wwwwgcgcmmmm\}.$$

Besides, some interesting findings were discovered by Head such as persistent, null-context, uniform splicing system and also strictly locally testable (SLT) language. These types of splicing systems have contributed to the exploring of new properties to the existing splicing systems. As an example, an in-depth study on the characterization of persistent splicing system has shown a relation with strictly locally testable language [2]. Gatterdam [12] has extended the study on persistent splicing system by introducing permanent splicing system that has quite similar meaning but with different approach. These two related definitions have been used by Yusof [13] in the study of the characterization of a non semi-simple splicing system. Karimi *et al.* [14] has explored these two systems in detail where some sufficient conditions were provided and some new definitions that is closely related to the original definitions such as self-closed, crossing-preserved and extended crossing-preserved were introduced. Recently, Mudabber *et al.* [15] has conducted a study on two stages splicing system in which its properties are explored in terms of persistency and permanency of the splicing system.

Since then, few types of splicing system were introduced. For instance, Mateescu *et al.* [16] has introduced a definition on simple splicing system which was then further studied by Laun based on the continuity aspect. An extension version of a simple and null-context splicing system has been introduced namely a semi-simple and semi-null splicing system [17] respectively. Then, it is proven that S_kH system is a simple splicing system if k = 1 which proves that the union of S_kH families is a family of SLT languages. Furthermore, a solid code is applied to the S_kH system to reduce it to the simple splicing system [18]. By the presence of various types of splicing system, some relations are established where simple splicing system $\subset$ semi-simple splicing system $\subset$ semi-null splicing system, and simple splicing system $\subset$ uniform splicing system $\subset$ S_k splicing system $\subset$ null-context splicing system [19]. Fong has conducted an intensive study to explore more properties and also some sufficient conditions of strictly locally testable language [18]. Some examples with different number of restriction enzymes were illustrated to show languages that are not strictly locally testable.

The varieties of the splicing system have been used to explore some properties on two different directions: either on the generation of languages or the preservation of biological traits in the splicing process. In the next section, recent advances on both the approaches in the splicing systems are presented.

RECENT ADVANCEMENT OF SPLICING SYSTEMS

A splicing language is produced by a splicing system. Head splicing system and its improved version, Y-G splicing system are normally chosen in studying the behavior of restriction enzyme on the dsDNA molecules. Based on the experiment, when the restriction enzymes react with a collection of DNA molecules, the splicing language is produced [2]. There are few types of splicing languages such as adult or inert, transient, and limit language [9]. In order to verify the results obtained from the splicing system such as to verify the model used, the existence of a specific language and its characteristics, a mathematical modelling of the splicing system is developed. As years passed by, several laboratory experiments have been conducted. For instance, Laun and Reddy in 1999 [20] has pioneered the first experiment on validating the accuracy of the Head model, which predicts the behavior of the corresponding biological process by using two restriction enzymes namely *Bgl*I and *Dra*III. Several other experimental works have also been performed such as Fong *et al.* in 2008 [21] to verify the mathematical model and to show the difference between adult and limit language by also using two restriction enzymes, namely *Aci*I and *Hpa*II. In 2013, Karimi [3] conducted an experiment to biologically validate the behavior of persistent splicing systems by using *CviQ*I and *Acc*65I. In early 2015, Ahmad *et al.* [22] conducted a laboratory experiment to verify the mathematical model and the existence of second order limit language [23] by using a restriction enzyme namely *Dpn*II. The model is validated if the generated DNA molecules through the experiment are same with the prediction in the model of the splicing language.

In the study of the splicing system that focuses on the generation of language, Paun and Pixton splicing system are used. Infinite cases which can be represented by the rules in Paun's or Pixton's model made it suitable to study the characterizations of the languages. In addition, splicing system that involves a set of finite axioms and rules produces only regular languages. Some restrictions are implemented to the splicing system to increase the computational power of the language from regular to the recursive enumerable language [7]. Karimi in [3] used fuzzy threshold languages approach to the persistent splicing system of Paun splicing system perspective to increase the computational power of splicing systems and to produce some non-regular languages. Besides that, Gan *et al.* [24] used weights where its properties are explored from some properties of group in term of automata to achieve higher computational power of the splicing system. Selvarajoo *et al.* [25] used probability instead to increase the computational power of the splicing system where probability is imposed to the splicing system.

CONCLUSION

As a conclusion, a relation between the formal language theory and the study of informational macromolecules is presented where some fundamental terms of formal language theory and some motivational background of DNA and its related subjects are given. Besides, five different models of the splicing system namely Head, Paun, Pixton, Goode-Pixton and Y-G splicing systems have been discussed and their differences shown. Some types of the splicing system and the relations that exist among them are presented. The current trends of the research which cover two directions of the research are provided to show the development of the splicing system throughout the years.

ACKNOWLEDGEMENTS

The first and third authors would like to thank Ministry of Higher Education (MOHE) Malaysia and Research Management Centre (RMC), Universiti Teknologi Malaysia (UTM) for the financial funding through Fundamental Research Grant Scheme (FRGS) Vote No. 4F590. The second author is indebted to the MOHE for his financial funding through MyBrain15 (MyPhD) scholarship.

REFERENCES

1. P. Linz, *An Introduction to Formal Languages and Automata* , Jones and Barlett Publisher, USA, 2006), pp. 1–25.
2. T. Head, B. Math, Biol. **49**, 737–759 (1987).
3. F. Karimi, "Mathematical Modelling of Splicing Persistent Splicing Systems in DNA Computing," Ph. D. thesis, Universiti Teknologi Malaysia, 2013.
4. G. Paun, G. Rozenberg and A. Salomaa, *DNA Computing New Paradigms* (Springer-Verlag, Germany, 1998), pp. 1–80.
5. I. E. Alcamo, *DNA Technology Awesome Skill*, 2nd Edition (Academic Press, USA, 2001), pp. 1–93.
6. R. H. Tamarin, *Principles of Genetics* (The Mac-Graw Hill Companies, USA, 2001), pp. 359–382.
7. Gh. Paun, *Discrete Appl. Math.* **70**, 57–79 (1996).
8. D. Pixton, *Discrete Appl. Math.* **69**, 101–124 (1996).
9. E. Goode and D. Pixton, "Splicing to the Limit," in *Aspects of Molecular Computing*, edited by N. Janoska, Gh. Paun and G. Rozenberg, (Springer Berlin, Heidelberg, New York, 2004), pp. 189–201.
10. Y. Yusof, N. H. Sarmin, T. E. Goode, M. Mahmud and W. H. Fong, "An Extension of DNA Splicing System," in *6th International Conference on Bio-Inspired Computing: Theories and Application (BIC-TA 2011)*, IEEE Conference Proceedings, edited by R. Abdullah *et al*. (IEEE, Piscataway, NJ, 2011), pp. 246–248.
11. P. Bonizzoni, C. Ferretti, G. Mauri and R. Zizza, Inform. Process Lett. **79**, 255–259 (2001).
12. R. W. Gatterdam, SIAM J. Comput. **21**, 507–520 (1992).
13. Y. Yusof, "DNA Splicing System Inspired by Bio-Molecular Operations," Ph. D. thesis, Universiti Teknologi Malaysia, 2011.
14. F. Karimi, N. H. Sarmin and W. H. Fong, Aust. J. Basic Appl. Sci. **5**, 20–24 (2011)
15. M. H. Mudaber, Y. Yusof and M. S. Mohamad, "Some Sufficient Conditions for Persistency and Permanency of Two Stages DNA Splicing Languages via Yusof-Goode Approach," in *Proceedings of the 21st National Symposium on Mathematical Sciences (SKSM21)*, AIP Conference Proceedings 1605, edited by M. T. Ismail, S. Ahmad and R. A. Rahman, (American Institute of Physics, Melville, NY, 2014), pp. 591–595.
16. A. Mateescu, Gh. Paun, G. Rozenberg and A. Salomaa, Discrete Appl. Math. **84**, 145–163 (1998).
17. T. E. G. Laun, "Constants and Splicing Systems," Ph.D. thesis, State University of New York, 1999.
18. W. H. Fong, "Modelling of Splicing Systems using Formal Language Theory," Ph. D. thesis, Universiti Teknologi Malaysia, 2008.
19. Y. Yusof, N. H. Sarmin, T. E. Goode, M. Mahmud and W. H. Fong, "Hierarchy of Certain Types of DNA Splicing Systems," in *International Conference Mathematical and Computational Biology 2011*, International Journal of Modern Physics: Conference Series 9, edited by K. A. M. Atan *et al.* (World Scientific, Hackensack, NJ, 2012), pp. 271–277.
20. E. Goode and D. Pixton, "Wet Splicing System," in *DIMAC Series in Discrete Mathematics and Theoretical Computer Science*, edited by H. Rubin and D. H. Wood (American Mathematical Society, Rhode Island, USA, 1999), pp. 73–83.

21. W. H. Fong and N. H. Sarmin, "Mathematical Modelling of Splicing Systems," in *Proceedings of the 1st International Conference on Natural Resources Engineering & Technology*, (2006), pp. 524–527.
22. M. A. Ahmad, N. H. Sarmin, M. F. A. Wahab, Y. Yusof and W. H. Fong, *J. Theor. Biol.* (submitted).
23. M. A. Ahmad, N. H. Sarmin, W. H. Fong and Y. Yusof, "An Extension of First Order Limit Language," in *Proceedings of the 3rd International Conference on Mathematical Sciences*, AIP Conference Proceedings 1602, edited by W. Z. W. Zain *et al.* (American Institute of Physics, Melville, NY, 2014), pp. 627–631.
24. S. Turaev, Y. S. Gan, M. Othman, N. H. Sarmin and W. H. Fong, "Weighted Splicing System," in *Advanced Methods for Computational Collective Intelligence*, edited by Z. Li *et al.* (Springer Berlin, Heidelberg, New York, 2012), pp. 416–424.
25. S. Turaev, M. Selvarajoo, M. S. Selamat, N. H. Sarmin and W. H. Fong, "Probabilistic Splicing System," in *Computational Intelligence and Intelligent Systems*, edited by N. T. Nguyen *et al.* (Springer Berlin, Heidelberg, New York, 2013), pp. 259–268.

Proceedings of "The 7th SEAMS-UGM Conference 2015"
Topic, pp. 126–132.

Long $\mathcal{U}$-exact Sequence of Homologies

Yudi Mahatma[1] and Intan Muchtadi-Alamsyah[2]

[1]*Department of Mathematics, Jakarta State University. Email : yudi_mahatma@unj.ac.id*
[2]*Department of Mathematics, Bandung Institute of Technology. Email :* ntan@math.itb.ac.id

Abstract. Given any short exact sequence of R-modules and R-module homomorphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the long exact sequence of homologies $\ldots \xrightarrow{\gamma_{n-1}} \mathrm{Ext}^n(M,A) \xrightarrow{\mathrm{Ext}^n(M,f)} \mathrm{Ext}^n(M,B) \xrightarrow{\mathrm{Ext}^n(M,g)} \mathrm{Ext}^n(M,C) \xrightarrow{\gamma_n} \ldots$ can always be formed for any R-module M. In this paper, the exactness would be generalized into U-exact case for any R-submodule U of C. The result is ultimately the generalization of the long exact sequence of homologies except that the exactness is failed to be preserved by the connecting morphisms.

Keywords : *exact sequence, homology, connecting morphism*

INTRODUCTION

In [1] Davvaz and Shabani-Solt have published the Generalization of Connecting Homomorphism Theorem and the Generalization of Exact Triangle Theorem. Those two theorems basically provide the idea of the generalization of long exact sequence of homologies, where the exactness here will be replaced by U-exact. Beside the short exact sequences, the construction of the connecting homomorphism will also be modified for general case as well. Further concepts related to this discussion can be found in the book [2].

SHORT $\mathcal{U}$-EXACT SEQUENCE OF HOMOLOGIES

Let R be an algebra, A, B, $C \in \mathrm{Mod}\,R$, and U be an R-submodule of C. Suppose that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short U-exact sequence, that is the sequence of R-modules and R-module homomorphisms which is 0-exact at A, U-exact at B, and 0-exact at C. Note that this means that f is one-to-one, $f(A) = g^{-1}(U)$, and g is onto. Such sequence is also called *the U-extension of A by C*. Now, for any $M \in \mathrm{Mod}\,R$ consider the sequence

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M,B) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M,C). \tag{1}$$

Suppose that $\alpha \in \mathrm{Hom}(M,A)$ with $\mathrm{Hom}(M,f)(\alpha) = f\alpha = 0$. Since f is one-to-one then α must be zero. This shows that $\mathrm{Hom}(M,f)$ is one-to-one. Now let $\beta \in \mathrm{Im}(\mathrm{Hom}(M,f))$. Suppose that $\beta = fh$ where $h \in \mathrm{Hom}(M,A)$. Notice that $\mathrm{Im}(g\beta) = \mathrm{Im}(gfh) \subseteq U$. Hence $\mathrm{Hom}(M,g)(\beta) = g\beta \in \mathrm{Hom}(M,U)$. This gives $\mathrm{Im}(\mathrm{Hom}(M,f)) \subseteq (\mathrm{Hom}(M,g))^{-1}(\mathrm{Hom}(M,U))$.

Next, notice that if $\beta \in (\mathrm{Hom}(M,g))^{-1}(\mathrm{Hom}(M,U))$ then $\mathrm{Hom}(M,g)(\beta) = g\beta \in \mathrm{Hom}(M,U)$. Since $\mathrm{Im}(g\beta) \subseteq U$ then $\mathrm{Im}(\beta) \subseteq g^{-1}(U) = f(A)$. Therefore, for every $m \in M$ there exists a unique element $a_m \in A$ so that $f(a_m) = \beta(m)$. Define the morphism $\alpha \in \mathrm{Hom}(M,A)$ by $m \mapsto a_m$ for every $m \in M$.

Obviously, $\beta = f\alpha \in \mathrm{Im}\big(\mathrm{Hom}(M,f)\big)$. This gives $\big(\mathrm{Hom}(M,g)\big)^{-1}\big(\mathrm{Hom}(M,U)\big) \subseteq \mathrm{Im}\big(\mathrm{Hom}(M,f)\big)$. From the preceding result, it can be concluded that $\mathrm{Im}\big(\mathrm{Hom}(M,f)\big) = \big(\mathrm{Hom}(M,g)\big)^{-1}\big(\mathrm{Hom}(M,U)\big)$. This shows that the sequence (1) is 0-exact at $\mathrm{Hom}(M,A)$ and $\mathrm{Hom}(M,U)$-exact at $\mathrm{Hom}(M,B)$.

Let $\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ be a projective resolution of M where P_0 is a free module generated by elements of M and d_0 is a natural map, P_1 is a free module generated by elements of $\ker(d_0)$ and d_1 is a natural map, P_2 is a free module generated by elements of $\ker(d_1)$ and d_2 is a natural map, and so forth. Now, for any $n \in \mathrm{N}$ and any module X define $\mathrm{Ext}^n(M,X) = \ker\big(\mathrm{Hom}(d_{n+1},X)\big)\big/\mathrm{Im}\big(\mathrm{Hom}(d_n,X)\big)$ and for any morphism $h: X \to Y$ define a map $\mathrm{Ext}^n(M,h): \mathrm{Ext}^n(M,X) \to \mathrm{Ext}^n(M,Y)$ by $[x] \mapsto [hx]$ for every $x \in \ker\big(\mathrm{Hom}(d_{n+1},X)\big)$. This map is well defined since $[x_1] = [x_2]$ implies $x_1 - x_2 \in \mathrm{Im}\big(\mathrm{Hom}(d_n,X)\big)$ and hence there exists a morphism $\delta: P_{n-1} \to X$ such that $x_1 - x_2 = \delta d_n$. Since $h\delta: P_{n-1} \to Y$ then $hx_1 - hx_2 = h(x_1 - x_2) = h\delta d_n \in \mathrm{Im}\big(\mathrm{Hom}(d_n,Y)\big)$ or, equivalently, $[hx_1] = [hx_2]$.

Now, consider the sequence

$$\mathrm{Ext}^n(M,A) \xrightarrow{\mathrm{Ext}^n(M,f)} \mathrm{Ext}^n(M,B) \xrightarrow{\mathrm{Ext}^n(M,g)} \mathrm{Ext}^n(M,C). \tag{2}$$

Consider the R-module $\mathrm{Ext}^n(M,U)$. Define a map $\theta: \mathrm{Ext}^n(M,U) \to \mathrm{Ext}^n(M,C)$ by $[z] \mapsto [z]$ for every $z \in \ker\big(\mathrm{Hom}(d_{n+1},U)\big) \subseteq \ker\big(\mathrm{Hom}(d_{n+1},C)\big)$. This map is well defined for $\mathrm{Im}\big(\mathrm{Hom}(d_n,U)\big) \subseteq \mathrm{Im}\big(\mathrm{Hom}(d_n,C)\big)$. Now, if $[z] \mapsto [z] = 0$ then $z \in \mathrm{Im}\big(\mathrm{Hom}(d_n,C)\big)$. Hence $z = \lambda d_n$ for some morphism $\lambda: P_{n-1} \to C$. In the other side, $z \in \ker\big(\mathrm{Hom}(d_{n+1},U)\big)$ and hence $z: P_n \to U$. Then $\lambda: P_{n-1} \to U$. Hence $z \in \mathrm{Im}\big(\mathrm{Hom}(d_n,U)\big)$. This shows that $\ker(\theta) = 0$ and hence θ is an injection. By this injection, $\mathrm{Ext}^n(M,U)$ can be regarded as an R-submodule of $\mathrm{Ext}^n(M,C)$. Otherwise, every class $[z] \in \mathrm{Ext}^n(M,C)$ with $z: P_n \to U$ can be considered as element of $\mathrm{Ext}^n(M,U)$.

Now let $[b] \in \mathrm{Im}\big(\mathrm{Ext}^n(M,f)\big)$. Suppose that $[b] = \mathrm{Ext}^n(M,f)([a]) = [fa]$. Notice that $\mathrm{Ext}^n(M,g)([b]) = [gfa]$ with $gfa: P_n \to U$. So, $[gfa] \in \mathrm{Ext}^n(M,U)$. Therefore, $\mathrm{Im}\big(\mathrm{Ext}^n(M,f)\big) \subseteq \mathrm{Ext}^n(M,g)^{-1}\big(\mathrm{Ext}^n(M,U)\big)$.

Conversely, if $[b] \in \mathrm{Ext}^n(M,g)^{-1}\big(\mathrm{Ext}^n(M,U)\big)$ then $\mathrm{Ext}^n(M,g)([b]) = [gb] \in \mathrm{Ext}^n(M,U)$. Hence, $gb: P_n \to U$. Therefore, for every $p \in P_n$, $b(p) \in g^{-1}(U) = f(A)$. So, for every $p \in P_n$ there exists a unique element $a_p \in A$ such that $b(p) = f(a_p)$. Define a morphism $a: P_n \to A$ by $p \mapsto a_p$ for every $p \in P_n$. Obviously, $b = fa$ and hence $[b] = [fa]$. This shows that $[b] \in \mathrm{Im}\big(\mathrm{Ext}^n(M,f)\big)$. Therefore, $\mathrm{Ext}^n(M,g)^{-1}\big(\mathrm{Ext}^n(M,U)\big) \subseteq \mathrm{Im}\big(\mathrm{Ext}^n(M,f)\big)$. From the preceding result, it can be concluded that $\mathrm{Im}\big(\mathrm{Ext}^n(M,f)\big) = \mathrm{Ext}^n(M,g)^{-1}\big(\mathrm{Ext}^n(M,U)\big)$. This shows that the sequence (2) is $\mathrm{Ext}^n(M,U)$-exact at $\mathrm{Ext}^n(M,B)$.

The results already obtained can be summarized in the following lemma :

Lemma 1 Given short U-exact sequence of R-modules dan R-module homomorphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, for any $M \in \mathrm{Mod}\,R$ the sequence $0 \to \mathrm{Hom}(M,A) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M,B) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M,C)$ is 0-exact at $\mathrm{Hom}(M,A)$ and

$\mathrm{Hom}(M,U)$-exact at $\mathrm{Hom}(M,B)$, and for any $n\in\mathrm{N}$ the sequence $\mathrm{Ext}^n(M,A)\xrightarrow{\mathrm{Ext}^n(M,f)}\mathrm{Ext}^n(M,B)\xrightarrow{\mathrm{Ext}^n(M,g)}\mathrm{Ext}^n(M,C)$ is $\mathrm{Ext}^n(M,U)$-exact at $\mathrm{Ext}^n(M,B)$.

CONNECTING MORPHISM

Let $h\in\mathrm{Hom}(M,C)$. Consider the following diagram :

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \to & 0 \\ & & & & & & \downarrow h & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \end{array}$$

Since P_0 is free and g is onto then it is always possible to construct a morphism $\beta_h : P_0 \to B$ such that $\mathrm{Im}(g\beta_h - hd_0)\subseteq U$. Now construct the morphism $\alpha_h : P_1 \to A$ as follows : For every $p\in P_1$, since $hd_0d_1(p)=0$ then $g\beta_h d_1(p)\in U$ or, equivalently, $\beta_h d_1(p)\in g^{-1}(U)=f(A)$. Hence $\beta_h d_1(p)=f(a_p)$ for a unique element a_p of A. Define $\alpha_h(p)=a_p$. Then $f\alpha_h=\beta_h d_1$. Therefore, $f\alpha_h d_2=\beta_h d_1 d_2=0$. Since f is one-to-one then $\alpha_h d_2=0$. Hence, $\alpha_h\in\ker(\mathrm{Hom}(d_2,A))$.

Clearly, the construction of α_h above depends on the choice of β_h. However, it can be shown that any two morphism α_h and α_h' constructed as above are congruent modulo $\mathrm{Im}(\mathrm{Hom}(d_1,A))$. Consider the following diagram :

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \to & 0 \\ & & \alpha_h\downarrow\downarrow\alpha_h' & & \beta_h\downarrow\downarrow\beta_h' & & \downarrow h & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \end{array}$$

where $\mathrm{Im}(g\beta_h - hd_0)\subseteq U$ and $\mathrm{Im}(g\beta_h' - hd_0)\subseteq U$, and $f\alpha_h=\beta_h d_1$ and $f\alpha_h'=\beta_h' d_1$. Notice first that $\mathrm{Im}(g(\beta_h-\beta')_h)\subseteq U$. Hence, for every $p\in P_0$, $(\beta_h-\beta_h')(p)\in g^{-1}(U)=f(A)$. Therefore, for every $p\in P_0$ there exists a unique element $a_p\in A$ such that $(\beta_h-\beta_h')(p)=f(a_p)$. Define the morphism $\delta:P_0\to A$ by $x\mapsto a_x$ for every $x\in P_0$. Recall that for every $p\in P_1$, $f(\alpha_h-\alpha_h')(p)=(\beta_h-\beta_h')d_1(p)$. Let $d_1(p)=x$. Then $f(\alpha_h-\alpha_h')(p)=(\beta_h-\beta_h')(x)=f(a_x)=f\delta(x)=f\delta d_1(p)$. Therefore, $f(\alpha_h-\alpha_h')=f\delta d_1$ and since f is one-to-one then $\alpha_h-\alpha_h'=\delta d_1$ and hence $\alpha_h=\alpha_h'+\delta d_1$. Since $\delta d_1\in\mathrm{Im}(\mathrm{Hom}(d_1,A))$ then $\alpha_h\equiv\alpha_h' \bmod \mathrm{Im}(\mathrm{Hom}(d_1,A))$.

From the paragraph above it is clear that every $h\in\mathrm{Hom}(M,C)$ can be mapped to a unique equivalence class $[\alpha_h]\in\ker(\mathrm{Hom}(d_2,A))/\mathrm{Im}(\mathrm{Hom}(d_1,A))=\mathrm{Ext}^1(M,A)$. Let $\gamma_0:\mathrm{Hom}(M,C)\to\mathrm{Ext}^1(M,A)$ be such morphism. Consider the following sequence

$$\mathrm{Hom}(M,B)\xrightarrow{\mathrm{Hom}(M,g)}\mathrm{Hom}(M,C)\xrightarrow{\gamma_0}\mathrm{Ext}^1(M,A)\xrightarrow{\mathrm{Ext}^1(M,f)}\mathrm{Ext}^1(M,B). \tag{3}$$

Let $c\in\ker(\gamma_0)$. Choose $\beta_c:P_0\to B$ such that $cd_0=g\beta_c$. Next, use β_c to construct $\alpha_c\in\ker(\mathrm{Hom}(d_2,A))$ that satisfies $f\alpha_c=\beta_c d_1$. Since $c\in\ker(\gamma_0)$ then $[\alpha_c]=0$ or, equivalently, $\alpha_c=\delta d_1$ for some morphism $\delta:P_0\to A$. Clearly, $(\beta_c-f\delta)d_1=0$. Now define the map $b:M\to B$ as follows : For every $m\in M$, if $m=d_0(q)$ set $b(m)=(\beta_c-f\delta)(q)$. Notice that if $m=d_0(q_1)=d_0(q_2)$ then

$q_1 - q_2 \in \ker(d_0) = \mathrm{Im}(d_1)$. Consequently, $(\beta_c - f\delta)(q_1 - q_2) = 0$. Hence the map b is well defined. Next, notice that for every $m \in M$, if $m = d_0(q)$ then $gb(m) = g\beta_c(q) - gf\delta(q) = cd_0(q) - gf\delta(q) = c(m) - gf\delta(q)$. Since $\mathrm{Im}(gf) \subseteq U$ then $\mathrm{Im}(c - gb) \subseteq U$.

Conversely, suppose that $c \in \mathrm{Hom}(M,C)$ such that there exists a morphism $b : M \to B$ with $\mathrm{Im}(c - gb) \subseteq U$. Denote bd_0 by β_c. Hence $\mathrm{Im}(cd_0 - g\beta_c) = \mathrm{Im}((c - gb)d_0) \subseteq U$. Therefore, β_c can be used to construct $\alpha_c \in \ker(\mathrm{Hom}(d_2, A))$ that satisfies $f\alpha_c = \beta_c d_1 = bd_0 d_1 = b0 = 0$. Since f is one-to-one then $\alpha_c = 0$ and hence $\gamma_0(c) = [\alpha_c] = 0$. Therefore, $c \in \ker(\gamma_0)$.

Now, for any module K, any morphism $h : X \to Y$, and any submodule Z of Y, define $[(K,h) : Z] = \{y \in \mathrm{Hom}(K,Y) \mid \exists x \in \mathrm{Hom}(K,X) \ni \mathrm{Im}(y - hx) \subseteq Z\}$. The last two paragraphs above show that $\ker(\gamma_0) = [(M,g) : U]$. Note that $[(M,g) : 0] = \mathrm{Im}(\mathrm{Hom}(M,g))$. Thus, in case $U = 0$, $\ker(\gamma_0) = \mathrm{Im}(\mathrm{Hom}(M,g))$ as it has to be. But in general, $\mathrm{Im}(\mathrm{Hom}(M,g)) \subseteq \ker(\gamma_0)$.

Now let $[z] \in \ker(\mathrm{Ext}^1(M,f))$. Since $[fz] = 0$ then $fz \in \mathrm{Im}(\mathrm{Hom}(d_1, B))$. Hence there exists a morphism $\delta : P_0 \to B$ such that $fz = \delta d_1$. Construct the morphism $c : M \to C/U$ as follows : For every $m \in M$, if $m = d_0(y)$ set $c(m) = g\delta(y) + U = \pi g\delta(y)$ where $\pi : C \to C/U$ is a natural projection. Notice that if $m = d_0(y_1) = d_0(y_2)$ then $y_1 - y_2 \in \ker(d_0) = \mathrm{Im}(d_1)$. Consequently, $g\delta(y_1 - y_2) \in U$. Hence the map c is well defined and there exists a diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \to & 0 \\ & & \downarrow z & & \downarrow \delta & & \downarrow c & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{\pi g} & C/U & \to & 0 \end{array} \qquad (4)$$

with $cd_0 = \pi g\delta$ and $fz = \delta d_1$.

Conversely, suppose that $[z] \in \mathrm{Ext}^1(M,A)$ and there exists a morphism $c : M \to C/U$ and a morphism $\delta : P_0 \to B$ such that $cd_0 = \pi g\delta$ and $fz = \delta d_1 \in \mathrm{Im}(\mathrm{Hom}(d_1, B))$. Then $[fz] = 0$ and hence $[z] \in \ker(\mathrm{Ext}^1(M,f))$.

The last two paragraphs above show that an equivalence class $[z] \in \mathrm{Ext}^1(M,A)$ is in $\ker(\mathrm{Ext}^1(M,f))$ if and only if there exists the commutative diagram (4) for some morphism $c : M \to C/U$ and some morphism $\delta : P_0 \to B$. Note that in case $U = 0$, $C/U = C$ and $\pi = 1_C$. Hence in those case, $\ker(\mathrm{Ext}^1(M,f)) = \mathrm{Im}(\gamma_0)$ as it has to be. But, in general $\mathrm{Im}(\gamma_0) \subseteq \ker(\mathrm{Ext}^1(M,f))$.

The result above shows that the sequence (3) is not necessarily exact but it is true that it must be a complex, that is a sequence $\cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$ with $\mathrm{Im}(d_{k+1}) \subseteq \ker(d_k)$. The summary is presented by the following lemma :

Lemma 2 Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short U-exact sequence of R-modules dan R-module homomorphisms and $M \in \mathrm{Mod}\,R$. Then the connecting morphism $\gamma_0 : \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,A)$ has properties that $\ker(\gamma_0) = \{c \in \mathrm{Hom}(M,C) \mid \exists b \in \mathrm{Hom}(M,B) \ni \mathrm{Im}(c - gb) \subseteq U\}$ and $\mathrm{Im}(\gamma_0) \subseteq \ker(\mathrm{Ext}^1(M,f))$.

Now, for any $n \in \mathrm{N}$ and $z \in \ker(\mathrm{Hom}(d_{n+1}, C))$ consider the following diagram :

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{d_{n+2}} & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & \cdots & \xrightarrow{d_0} & M & \to & 0 \\ & & & & & \square\ z & & & & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 & & \end{array}$$

Since P_n is free and g is onto then it is always possible to construct a morphism $\delta_z : P_n \to B$ such that $\mathrm{Im}(g\delta_z - z) \subseteq U$. Now construct the morphism $\varepsilon_z : P_{n+1} \to A$ as follows : For every $p \in P_{n+1}$, since $zd_{n+1}(p) = 0$ then $g\delta_z d_{n+1}(p) \in U$ or, equivalently, $\delta_z d_{n+1}(p) \in g^{-1}(U) = f(A)$. Hence $\delta_z d_{n+1}(p) = f(a_p)$ for a unique element $a_p \in A$. Define $\varepsilon_z(p) = a_p$. Then $f\varepsilon_z = \delta_z d_{n+1}$. Therefore, $f\varepsilon_z d_{n+2} = \delta_z d_{n+1} d_{n+2} = 0$. Since f is one-to-one then $\varepsilon_z d_{n+2} = 0$. Hence, $\varepsilon_z \in \ker(\mathrm{Hom}(d_{n+2}, A))$.

Clearly, the construction of ε_z above depends on the choice of δ_z. However, it can be shown that any two morphisms ε_z and ε'_z constructed as above are congruent modulo $\mathrm{Im}(\mathrm{Hom}(d_{n+1}, A))$. Consider the following diagram :

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{d_{n+2}} & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & \cdots & \xrightarrow{d_0} & M & \to & 0 \\ & & \varepsilon_z \downarrow\downarrow \varepsilon'_z & & \delta_z \downarrow\downarrow \delta'_z & \square & z & & & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 & & \end{array}$$

where $\mathrm{Im}(g\delta_z - z) \subseteq U$ and $\mathrm{Im}(g\delta'_z - z) \subseteq U$, and $f\varepsilon_z = \delta_z d_{n+1}$ and $f\varepsilon'_z = \delta'_z d_{n+1}$. Notice first that $\mathrm{Im}(g(\delta_z - \delta')_z) \subseteq U$. Hence, for every $p \in P_n$, $(\delta_z - \delta'_z)(p) \in g^{-1}(U) = f(A)$. Therefore, for every $p \in P_n$ there exists a unique element $a_p \in A$ such that $(\delta_z - \delta'_z)(p) = f(a_p)$. Define the morphism $\eta : P_n \to A$ by $x \mapsto a_x$ for every $x \in P_n$. Recall that for every $p \in P_n$, $f(\varepsilon_z - \varepsilon'_z)(p) = (\delta_z - \delta'_z)d_{n+1}(p)$. Let $d_{n+1}(p) = x$. Then $f(\varepsilon_z - \varepsilon'_z)(p) = (\delta_z - \delta'_z)(x) = f(a_x) = f\eta(x) = f\eta d_{n+1}(p)$. Therefore, $f(\varepsilon_z - \varepsilon'_z) = f\eta d_{n+1}$ and since f is one-to-one then $\varepsilon_z - \varepsilon'_z = \eta d_{n+1}$ and hence $\varepsilon_z = \varepsilon'_z + \eta d_{n+1}$. Since $\eta d_{n+1} \in \mathrm{Im}(\mathrm{Hom}(d_{n+1}, A))$ then $\varepsilon_z \equiv \varepsilon'_z \bmod \mathrm{Im}(\mathrm{Hom}(d_{n+1}, A))$.

From the paragraph above it is clear that every $z \in \ker(\mathrm{Hom}(d_{n+1}, C))$ can be mapped to a unique equivalence class $[\varepsilon_z] \in \ker(\mathrm{Hom}(d_{n+2}, A)) / \mathrm{Im}(\mathrm{Hom}(d_{n+1}, A)) = \mathrm{Ext}^{n+1}(M, A)$. Now let $\varphi : P_{n-1} \to C$. Notice that $z + \varphi d_n \in \ker(\mathrm{Hom}(d_{n+1}, C))$. Consider the morphisms $\delta_{z+\varphi d_n}$ and $\varepsilon_{z+\varphi d_n}$ defined as above. It is easy to check that $\mathrm{Im}(g(\delta_z - \delta_{z+\varphi d_n}) + \varphi d_n) \subseteq U$. Now, since P_{n-1} is free and g is onto then there exists a morphism $h : P_{n-1} \to B$ such that $\varphi = gh$. Notice that $\mathrm{Im}(\delta_z - \delta_{z+\varphi d_n} + hd_n) \subseteq g^{-1}(U) = f(A)$. Now, $f(\varepsilon_z - \varepsilon_{z+\varphi d_n}) = (\delta_z - \delta_{z+\varphi d_n})d_{n+1} = (\delta_z - \delta_{z+\varphi d_n} + hd_n)d_{n+1}$ since $d_n d_{n+1} = 0$. Define $\psi : P_n \to A$ as follows : For every $x \in P_n$, since $(\delta_z - \delta_{z+\varphi d_n} + hd_n)(x) \in f(A)$ then there exists a unique element $a_x \in A$ such that $(\delta_z - \delta_{z+\varphi d_n} + hd_n)(x) = f(a_x)$. Define $\psi(x) = a_x$. Then, for every $p \in P_{n+1}$, $f(\varepsilon_z - \varepsilon_{z+\varphi d_n})(p) = (\delta_z - \delta_{z+\varphi d_n} + hd_n)d_{n+1}(p)$. Suppose that $d_{n+1}(p) = x$. Then $f(\varepsilon_z - \varepsilon_{z+\varphi d_n})(p) = (\delta_z - \delta_{z+\varphi d_n} + hd_n)(x) = f(a_x) = f\psi(x) = f\psi d_{n+1}(p)$. Therefore, $f(\varepsilon_z - \varepsilon_{z+\varphi d_n}) = f\psi d_{n+1}$ and since f is one-to-one then $\varepsilon_z - \varepsilon_{z+\varphi d_n} = \psi d_{n+1} \in \mathrm{Im}(\mathrm{Hom}(d_{n+1}, A))$. Hence, $[\varepsilon_z] = [\varepsilon_{z+\varphi d_n}]$.

From the paragraph above it is clear that every equivalence class $[z] \in \ker(\mathrm{Hom}(d_{n+1}, C)) / \mathrm{Im}(\mathrm{Hom}(d_n, A)) = \mathrm{Ext}^n(M, C)$ can be mapped to a unique equivalence class $[\varepsilon_z] \in \mathrm{Ext}^{n+1}(M, A)$. Let $\gamma_n : \mathrm{Ext}^n(M, C) \to \mathrm{Ext}^{n+1}(M, A)$ be such morphism. Consider the following sequence

$$\mathrm{Ext}^n(M, B) \xrightarrow{\mathrm{Ext}^n(M, g)} \mathrm{Ext}^n(M, C) \xrightarrow{\gamma_n} \mathrm{Ext}^{n+1}(M, A) \xrightarrow{\mathrm{Ext}^{n+1}(M, f)} \mathrm{Ext}^{n+1}(M, B). \quad (5)$$

Let $[c]\in\ker(\gamma_n)$. Choose $\delta_c : P_n \to B$ such that $c = g\delta_c$. Next, use δ_c to construct $\varepsilon_c \in \ker\left(\mathrm{Hom}(d_{n+2}, A)\right)$ that satisfies $f\varepsilon_c = \delta_c d_{n+1}$. Since $[c]\in\ker(\gamma_n)$ then $[\varepsilon_c]=0$ or, equivalently, $\varepsilon_c = \eta d_{n+1}$ for some morphism $\eta : P_n \to A$. Clearly, $(\delta_c - f\eta)d_{n+1} = 0$ and hence $[\delta_c - f\eta]\in \mathrm{Ext}^n(M,B)$. Now, $\mathrm{Ext}^n(M,g)([\delta_c - f\eta]) = [g(\delta_c - f\eta)] = [c - gf\eta]$.

Conversely, suppose that $[c]\in\mathrm{Ext}^n(M,C)$ and there exists a morphism $\eta : P_n \to A$ such that $[c-gf\eta]\in\mathrm{Im}\left(\mathrm{Ext}^n(M,g)\right)$. Suppose that $[c-gf\eta]=[gb]$ where $[b]\in\mathrm{Ext}^n(M,B)$. Hence $\left[c-g(f\eta-b)\right]\in\mathrm{Im}\left(\mathrm{Hom}(d_n,C)\right)$ or, equivalently, $c-g(f\eta-b)=\lambda d_n$ for some morphism $\lambda : P_{n-1}\to C$. Set $\delta_{c-\lambda d_n} = f\eta - b$. Since $\mathrm{Im}\left(c-\lambda d_n - g\delta_{c-\lambda d_n}\right)=0\subseteq U$ then $\delta_{c-\lambda d_n}$ can be used to construct $\varepsilon_{c-\lambda d_n}\in\ker\left(\mathrm{Hom}(d_{n+2},A)\right)$ that satisfies $f\varepsilon_{c-\lambda d_n} = \delta_{c-\lambda d_n}d_{n+1} = (f\eta - b)d_{n+1} = f\eta d_{n+1}$ since $bd_{n+1}=0$. Since f is one-to-one then $\varepsilon_{c-\lambda d_n} = \eta d_{n+1}\in\mathrm{Im}\left(\mathrm{Hom}(d_{n+1},A)\right)$ and hence $\gamma_n([c]) = \gamma_n([c-\lambda d_n]) = [\varepsilon_{c-\lambda d_n}] = 0$. Therefore, $[c]\in\ker(\gamma_n)$.

The last two paragraphs above shows that an equivalence class $[c]\in\mathrm{Ext}^n(M,C)$ is in $\ker(\gamma_n)$ if and only if there exists a morphism $\eta : P_n \to A$ such that $[c-gf\eta]\in\mathrm{Im}\left(\mathrm{Ext}^n(M,g)\right)$. Remember that $\mathrm{Im}(gf)=U$. Hence, in case $U=0$, $[c]\in\mathrm{Ext}^n(M,C)$ is in $\ker(\gamma_n)$ if and only if $[c]\in\mathrm{Im}\left(\mathrm{Ext}^n(M,g)\right)$. Thus, in those case, $\ker(\gamma_n)=\mathrm{Im}\left(\mathrm{Ext}^n(M,g)\right)$ as it has to be. But in general, $\mathrm{Im}\left(\mathrm{Ext}^n(M,g)\right)\subseteq\ker(\gamma_n)$.

Now let $[a]\in\ker\left(\mathrm{Ext}^{n+1}(M,f)\right)$. Since $[fa]=0$ then $fa\in\mathrm{Im}\left(\mathrm{Hom}(d_{n+1},B)\right)$. Hence there exists a morphism $\psi : P_n \to B$ such that $fa = \psi d_{n+1}$. Define the morphism $c : P_n \to C/U$ by $p\mapsto g\psi(p)+U = \pi g\psi(p)$ for every $p\in P_n$, where $\pi : C\to C/U$ is a natural projection. Then there exists a diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d_{n+2}} & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \xrightarrow{d_0} & M & \to & 0 \\
 & & \downarrow a & & \downarrow\psi & \square & c & & & & & & \\
0 & \to & A & \xrightarrow{f} & B & \xrightarrow{\pi g} & C/U & \to & 0 & & & &
\end{array} \qquad (6)$$

with $cd_0 = \pi g\psi$ and $fa = \psi d_{n+1}$.

Conversely, suppose that $[a]\in\mathrm{Ext}^{n+1}(M,A)$ and there exists a morphism $c : P_n \to C/U$ and a morphism $\psi : P_n \to B$ such that $c = \pi g\psi$ and $fa = \delta d_{n+1}\in\mathrm{Im}\left(\mathrm{Hom}(d_{n+1},B)\right)$. Then $[fa]=0$ and hence $[a]\in\ker\left(\mathrm{Ext}^{n+1}(M,f)\right)$.

The last two paragraphs above shows that an equivalence class $[a]\in\mathrm{Ext}^{n+1}(M,A)$ is in $\ker\left(\mathrm{Ext}^{n+1}(M,f)\right)$ if and only if there exists the commutative diagram (6) for some morphism $c : P_n \to C/U$ and some morphism $\psi : P_n \to B$. Note that in case $U=0$, $C/U = C$ and $\pi = 1_C$. Hence, in those case, $\ker\left(\mathrm{Ext}^{n+1}(M,f)\right) = \mathrm{Im}(\gamma_n)$ as it has to be. But, in general $\mathrm{Im}(\gamma_n)\subseteq\ker\left(\mathrm{Ext}^{n+1}(M,f)\right)$.

The result above shows that the sequence (5) is not necessarily exact but it is true that it must be a complex. The summary is presented by the following lemma :

Lemma 3 Let $0\to A\xrightarrow{f}B\xrightarrow{g}C\to 0$ be a short U-exact sequence of R-modules dan R-module homomorphisms and $M\in\mathrm{Mod}\,R$. Then for every $n\in\mathrm{N}$, the connecting morphism $\gamma_n : \mathrm{Ext}^n(M,C)\to\mathrm{Ext}^{n+1}(M,A)$ has properties that $\ker(\gamma_n) = \left\{c\in\mathrm{Ext}^n(M,C)\,\middle|\,\exists\,\eta\in\mathrm{Hom}(P_n,A)\ni[c-gf\eta]\in\mathrm{Im}\left(\mathrm{Ext}^n(M,g)\right)\right\}$ and $\mathrm{Im}(\gamma_n)\subseteq\ker\left(\mathrm{Ext}^{n+1}(M,f)\right)$.

Next is the main result of this paper.

Theorem 1 Given short U-exact sequence of R-modules dan R-module homomorphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, for any $M \in \operatorname{Mod} R$ there always exists a long $\mathcal{U}$-exact sequence of homologies

$$
\begin{array}{ccccccc}
0 & \to & \operatorname{Hom}(M,A) & \xrightarrow{\operatorname{Hom}(M,f)} & \operatorname{Hom}(M,B) & \xrightarrow{\operatorname{Hom}(M,g)} & \operatorname{Hom}(M,C) \\
 & & & & & & \downarrow \gamma_0 \\
 & & \operatorname{Ext}^1(M,C) & \xleftarrow{\operatorname{Ext}^1(M,g)} & \operatorname{Ext}^1(M,B) & \xleftarrow{\operatorname{Ext}^1(M,f)} & \operatorname{Ext}^1(M,A) \\
 & & \downarrow \gamma_1 & & & & \\
 & & \operatorname{Ext}^2(M,A) & \xrightarrow{\operatorname{Ext}^2(M,f)} & \operatorname{Ext}^2(M,B) & \xrightarrow{\operatorname{Ext}^2(M,g)} & \operatorname{Ext}^2(M,C) \xrightarrow{\gamma_2} \ \ldots
\end{array}
$$

whose properties :

1. The sequence is 0-exact at $\operatorname{Hom}(M,A)$ and $\operatorname{Hom}(M,U)$-exact at $\operatorname{Hom}(M,B)$,
2. the sequence $\operatorname{Hom}(M,B) \xrightarrow{\operatorname{Hom}(M,g)} \operatorname{Hom}(M,C) \xrightarrow{\gamma_0} \operatorname{Ext}^1(M,A) \xrightarrow{\operatorname{Ext}^1(M,f)} \operatorname{Ext}^1(M,B)$ is a complex,
3. for every $n \in \mathrm{N}$, the sequence is $\operatorname{Ext}^n(M,U)$-exact at $\operatorname{Ext}^n(M,B)$, and
4. for every $n \in \mathrm{N}$, the sequence $\operatorname{Ext}^n(M,B) \xrightarrow{\operatorname{Ext}^n(M,g)} \operatorname{Ext}^n(M,C) \xrightarrow{\gamma_n} \operatorname{Ext}^{n+1}(M,A) \xrightarrow{\operatorname{Ext}^{n+1}(M,f)} \operatorname{Ext}^{n+1}(M,B)$ is a complex.

PROOF The properties 1 and 3 are exactly the Lemma 1, and the properties 2 and 4 are exactly Lemma 2 and Lemma 3, respectively.

CONCLUSION

The long exact sequence of homologies can be generalized for U-exact case of the short sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. Generally, the sequence obtained is not exact but it must be a complex. The Hom and Ext morphisms preserve the exactness while the connecting morphism does not.

REFERENCES

1. Davvaz, B. and H. Shabani-Solt, 2002, *A Generalization of Homological Algebra,* J. Korean Math Soc. **39**, No 6, 881-898.
2. Rotman, J. J., 1979, *An Introduction to Homological Algebra*, Academic Press, New York-London.